Research Center for Artificial Intelligence
Department of Computer Science
University of Craiova, Romania

Nicolae Ţăndăreanu

Lecture Notes on Universal Algebra.

Basic Concepts of Peano Algebras and Lattices.

Research Report 301

2006

Table of Contents

Preface

A great number of research works and practical implementations have confirmed the interest of mathematicians and computer scientists in developing and applying the methods of universal algebra. The aim of universal algebra is to extract the common elements (concepts, results and constructions) of algebraic structures, which can be identified as central entities for a large variety of the basic algebraic structures: group, semigroup, monoid, ring, module, lattice, semilattice, Boolean algebra and so on.

In general, an algebraic structure uses at least one algebraic operation. Two kinds of such operations can be considered: partial operations and total operations. The subject treated in this volume includes both the first and the second kind of operation. There are several motivations for this subject:

- Mathematicians are interested to study the common constructions of some mathematical structures in order to distinct them from the specific constructions.
- The computer scientists are interested to study this domain because a lot of problems implemented on computers were modeled by using the tools of universal algebra.
- The interest for the study of partial algebras can be explained by the fact that specific problems connected by partial computations have been encountered in mathematics (partial division for integer numbers, partial subtraction of natural numbers, the inverse of a matrix in numerical analysis) and are encountered today in mathematics of computer science (partial recursive mappings, partial computations in automata theory, answer function of the knowledge representation and processing systems, logical computation with respect to the semantics of logic programs, partial morphisms in algebraic models for knowledge representation, partial computations in the domain of the reasoning modeling, partial computations in the computability domain). As a consequence of this interest several new structures are developed today: test algebras, orthomodular algebras, many-sorted algebras and so on.

We give now a short description of the contents of the present volume. Chapter 1 provides the fundamentals of ordered sets (partial mappings, partial order, dual order, duality principle) and partial Σ-algebras (morphisms, subalgebras, free generated algebras). Chapter 2 deals with the Peano Σ-algebras (the construction of such structures, the isomorphism of the Peano algebras generated by the same set). Chapter 3 provides the basic properties of a fundamental structure named *lattice*. We describe the concepts of *lattice in the sense of Ore* and *the lattice in the sense of Dedekind*. We prove they are equivalent structures. Finally the concept of Boolean algebra is defined in a concise manner as a specialized form of lattice. In Chapter 4 we treat some distinctive problems concerning the results specified in the previous chapters. In the final of this volume we included Chapter 5, which should be viewed as a chapter describing some possible ideas to initiate the reader in a research activity allowing to obtain new results in these domains. Several open problems are specified here.

The most properties included in this volume are accompanied by their proofs. We consider that following these proofs the reader is able to obtain a better understanding of the concepts. On the other hand we intended to disengage the reader from an additional activity if the proofs are requested.

Besides the general interest to study the domain of universal algebra there is a specific feature of this aspect which is described by a "local" interest. The aim of the present volume is twofold and reflects the author's interest:

- to give a concise mathematical background for the course of "Computability and Deduction in Artificial Intelligence" (first year, Master in Computer Science, University of Craiova);
- to offer an initial study for Ph.D. students in informatics.

We relieve the fact that only the concepts and results used for a better understanding of the mentioned course are presented in this volume. The following topics of this course benefit of all results:

- the computability of the answer mapping of the knowledge systems based on inheritance;
- knowledge modeling by labeled stratified graphs;
- knowledge modeling by semantic schemas.

The text included in this volume is not an encyclopedic one. We hope the reader will find in this volume a concise description of the main results and a helpful literature to apply the algebraic methods in knowledge representation but not only.

It is not possible to finalize this preface before to thank Professor Sergiu Rudeanu. Under his guidance I obtained the first research results in this domain.

This volume is addressed to graduate students (master in computer science, Ph.D. students in informatics) and, in general, to each person which intend to apply universal algebras in various domains.

Nicolae Ţăndăreanu June 2006

1. Basic concepts of universal algebra

In this chapter we describe some of the basic concepts of universal algebra. The main concepts treated here are the following: partial order and partially ordered sets, duality principle, partial algebra, closed sets and partial subalgebras, algebraic induction, morphisms of partial algebras, free generated algebras.

1.1 Notations, basic concepts and results

We denote by N the set $\{0, 1, 2, \ldots\}$ of all natural numbers. We recall the basic concepts and notations from set theory.

We consider a non empty set A. The notation $B \subseteq A$ specifies that B is a subset of A. If $B \subseteq A$ and $B \neq A$ then we write $B \subset A$. The empty set is denoted by $\emptyset$.

The Cartesian product of the sets $X_1, X_2, \ldots, X_n$ is the set

$$X_1 \times X_2 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i, i = 1, \ldots, n\}$$

For a subset $X \subseteq X_1 \times X_2 \times \ldots \times X_n$ and $j \in \{1, \ldots, n\}$ we denote

$$pr_j X = \{x \in X_j \mid \exists (x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n) \in X\}$$

and this set is named the *projection* of X on the axis j.

We define $A^0 = \{\emptyset\}$ and for $n > 0$ we denote by A^n the Cartesian product $A \times \ldots \times A$ of n elements. By 2^A we denote the power set of A, i.e. the set of all subsets of A.

The words *mapping* and *function* are used as equivalent concepts. A mapping f from A to B is a set $f \subseteq A \times B$ such that

$$(x, y_1) \in f, (x, y_2) \in f \Longrightarrow y_1 = y_2$$

We denote $dom(f) = pr_1 f$ and this set is called the *domain of definition* of the mapping f. If $dom(f) \subset A$ then f is a *partial* mapping. Otherwise we say that f is a mapping *on* A.

The classical notation to indicate that f is a mapping from A to B is $f : A \longrightarrow B$. If $(x, y) \in f$ then we denote $y = f(x)$.
For a mapping $f : dom(f) \longrightarrow Y$ and $X \subseteq A$we denote

$$f(X) = \{y \mid \exists x \in X \cap dom(f) : f(x) = y\}$$

and this is *the image* of the set X by the mapping f. Particularly we denote $val(f) = f(dom(f))$.

If f is a (partial) mapping from A to B and $Y \subseteq B$ then we denote

$$f^{-1}(Y) = \{x \in dom(f) \mid f(x) \in Y\}$$

For a total mapping $f : A \longrightarrow B$ we have

$$X \subseteq A \Longrightarrow X \subseteq f^{-1}(f(X)) \tag{1.1}$$

$$Y \subseteq Z \subseteq B \Longrightarrow f^{-1}(Y) \subseteq f^{-1}(Z) \tag{1.2}$$

We write $f \prec g$ if $f : dom(f) \longrightarrow A$ and $g : dom(g) \longrightarrow A$ are two functions such that $dom(f) \subseteq dom(g)$ and $f(x) = g(x)$ for all $x \in dom(f)$. If this is the case, we say that f is a *restriction* of g.

If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are two mappings then we denote by $g \circ f$ the mapping $g \circ f : A \longrightarrow C$ defined by $g \circ f(x) = g(f(x))$. This is the *composition* operation or the *superposition* of two mappings.

We denote by 1_A the mapping $1_A : A \longrightarrow A$ defined by $1_A(x) = x$ for each $x \in A$. An useful property is given in the next proposition.

Proposition 1.1.1. *If $f : A \longrightarrow B$ and $g : B \longrightarrow A$ satisfy the identities*

$$g \circ f = 1_A, f \circ g = 1_B$$

then

1) *f and g are bijective mappings*
2) $g = f^{-1}$

Proof. If $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$ then $g(f(x_1)) = g(f(x_2))$, therefore $x_1 = x_2$. Thus the mapping f is injective. Let be $y \in B$. For $x = g(y)$ we have $f(x) = f(g(y)) = 1_B(y) = y$, therefore f is surjective. Similarly we prove that g is a bijective mapping. If $f(x) = y$ then $1_A(x) = g(f(x)) = g(y)$, i.e. $x = g(y)$. Conversely, if $x = g(y)$ then $f(x) = f(g(y)) = 1_B(y) = y$. Thus $f(x) = y$ if and only if $x = g(y)$. This establishes the relation $g = f^{-1}$. ■

A *binary relation* on the set A is a subset $\rho \subseteq A \times A$. In the vision of the usual relation $\leq$ between the real numbers we denote $x\rho y$ instead of $(x, y) \in \rho$. A binary relation ρ is a *partial order* if the following conditions are satisfied:

- Reflexivity: $x\rho x$ for all $x \in A$
- Antisymmetry: $x\rho y \wedge y\rho x \Longrightarrow x = y$ for all $x, y \in A$
- Transitivity: $x\rho y \wedge y\rho z \Longrightarrow x\rho z$ for all $x, y, z \in A$

If $x\rho y$ or $y\rho x$ then the elements x and y are called *comparable* elements. Otherwise x and y are *incomparable* elements. If, in addition of reflexivity, antisymmetry and transitivity we have $x\rho y$ or $y\rho x$ for every $x, y \in A$ then we say that ρ is a *total order* on A.

A *partially ordered set* or *poset* is a pair (A, ρ), where ρ is a partial order on the set A. If ρ is a total order then this pair is a *totally ordered set.*

For a subset $P \subseteq A$ an element $z \in A$ is a *lower bound* of P if $z\rho x$ for all $x \in P$. An element $z_0 \in A$ is the *greatest lower bound* of P if

- z_0 is a lower bound of P;
- for every lower bound z of P we have $z\rho z_0$.

We remark that although ρ is a partial order, if z_0 is the greatest lower bound of P and z is a lower bound of P then z_0 and z are *comparable elements* and moreover, we have $z\rho z_0$. Equivalently we say that z_0 is the *infimum* of P and we denote $z_0 = inf_\rho P$. If $z_0 = inf_\rho P$ and $z_0 \in P$ then z_0 is the *least* or the *first* element of P. The element $inf_\rho P$, if exists, is unique.

An element $u \in A$ is an *upper bound* of P if $x\rho u$ for all $x \in P$. An element $u_0 \in A$ is the *least upper bound* of P if

- u_0 is an upper bound of P;
- for every upper bound u of P we have $u_0\rho u$.

The same remark as in the case of $inf_\rho P$ can be relieved here: the least upper bound is comparable with each upper bound. Usually we say that the element u_0 above defined, if exists, is the *supremum* of the set P and we denote $u_0 = sup_\rho P$. If $u_0 = sup_\rho P$ and $u_0 \in P$ then u_0 is the *last* or the *greatest* element of P. The greatest element of a subset, if exists, is unique.

For a binary relation ρ on A we can define the *dual* relation, denoted by $\widetilde{\rho}$ and defined as follows:

$$x\widetilde{\rho}y \Longleftrightarrow y\rho x$$

Obviously the dual relation $\widetilde{\rho}$ of a partial order ρ is also a partial order.

Using the duality mentioned above we obtain *dual concepts.* For example, suppose α is a lower bound of the set $P \subseteq A$ in $(A, \widetilde{\rho})$. This means that $\alpha\widetilde{\rho}x$ for all $x \in P$. Equivalently we have $x\rho\alpha$ for every $x \in P$. Thus, a lower bound in $(A, \widetilde{\rho})$ is an *upper bound* in (A, ρ) and vice versa. Thus the concepts of lower bound and upper bound are dual concepts. Similarly, the least element and the greatest element are dual concepts.

From a sentence S stated in the terms of some domain we can obtain another statement $\widetilde{S}$ by replacing each concept by its dual. The statement $\widetilde{S}$ is named the *dual* of S. Particularly, this aspect can be relieved for theorems, i.e. for sentences that are proved in the corresponding domain. Based on the following principle we can save the proof of some theorems:

Proposition 1.1.2. *(Duality principle, Grätzer (1971))*
If T is a theorem in the theory of partially ordered sets then its dual $\widetilde{T}$ is also a theorem in the same domain.

For example, it is known the following theorem T: *The supremum of a subset, if exists, is unique.* The dual $\widetilde{T}$ is the sentence *The infimum of a subset, if exists, is unique* and this is a theorem in the theory of partially ordered sets. In virtue of the duality principle it is not necessary to give a proof of $\widetilde{T}$. As a matter of fact, the proof of $\widetilde{T}$ can be easy obtained by duality from the proof of T.

If does not exist any confusion, the index ρ from the notation $inf_\rho P$ or $sup_\rho P$ is omitted.

Example 1.1.1. Take the set $A = \{2, 3, 4, 6, 12\}$ and the relation "divides", $x\rho y$ if and only if x divides y. Consider $P = \{6, 12\}$. The elements 2 and 3 are lower bounds of P and they are incomparable elements. The greatest element of P is 12.

If (A, ρ) is a poset then we say that x *is covered* by y in A and we write $x\rho_c y$ if $x\rho y$, $x \neq y$ and from $x\rho z$ and $z\rho y$ we deduce $x = z$ or $z = y$. In other words, there isn't any element "between" x and y. Based on this relation we can represent any finite partially ordered set by a picture called *Hasse diagram.* In order to obtain such a diagram we proceed as follows: we draw any element of A by a circle and if $a\rho_c b$ then we draw the circle of b above the circle of a and then we join the two circles by a line segment. An example of such representation is given in Figure 1.1, where $A = \{0, a, b, c, 1\}$ and 0 is the least element, 1 is the greatest element, a and b are incomparable elements, b is covered by c, $inf\{a, c\} = 0$, $sup\{a, b\} = sup\{a, c\} = 1$ and so on. We observe that from a Hasse diagram we can rebuild the initial relation ρ by means of the sequences of covered elements and using also the reflexivity of the relation ρ. For example, we have $0\rho_c b$, $b\rho_c c$ and $c\rho_c 1$. It follows that $0\rho c$, $b\rho 1$ by such sequences of covered elements and then $b\rho b$ by reflexivity.

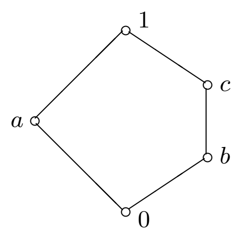


Fig. 1.1. Hasse diagram

A binary relation ρ on the set A is an *equivalence* relation if the following conditions are satisfied:

- Reflexivity: $x\rho x$ for all $x \in A$;
- Symmetry: $x\rho y \Longrightarrow y\rho x$ for all $x, y \in A$;

- Transitivity: $x\rho y \wedge y\rho z \Longrightarrow x\rho z$.

For an element $x \in A$ we define the *equivalence class* of x as follows:

$$[x]_\rho = \{y \in A \mid y\rho x\}$$

Proposition 1.1.3. (*Burris (1981)*)
If ρ is an equivalence relation on the set A then:

- *$x \in [x]_\rho$ for every $x \in A$;*
- *if $x\rho y$ then $[x]_\rho = [y]_\rho$;*
- *either $[x]_\rho \cap [y]_\rho = \emptyset$ or $[x]_\rho = [y]_\rho$.*

Proof. By the reflexivity property of ρ we have $x \in [x]_\rho$. Suppose $x\rho y$. By symmetry and transitivity of ρ we have $z\rho x$ if and only if $z\rho y$, therefore $[x]_\rho = [y]_\rho$. Suppose now that $[x]_\rho \cap [y]_\rho \neq \emptyset$ and take an element $z \in [x]_\rho \cap [y]_\rho$. It follows that $z\rho x$ and $z\rho y$ therefore $x\rho y$. Thus $[x]_\rho = [y]_\rho$. ■

1.2 Partial algebras: definitions and examples

The domain of universal algebras extracts and generalizes basic concepts and results from various algebraic structures. By this process not only an unitary theory is obtained but also the results can be successfully applied to new contexts. In this section the reader is familiarized with background concepts of this domain. The first concept is given in the next definition.

Definition 1.2.1. (*Burmeister (2002)*)
We consider a nonempty set A and a natural number $n \in N$. By an **n-ary partial operation** *on A we understand a partial mapping f from A^n to A. This means that $f : dom(f) \longrightarrow A$, where $dom(f) \subset A^n$. In the case when $dom(f) = A^n$ we say that f is an* **n-ary operation** *on A. The number n is called* **the arity of** *A.*

Let us consider an operation f of arity zero. We have two cases:

- if f is a partial operation then $dom(f) \subset A^0$, therefore $dom(f) = \emptyset$;
- if f is an operation then $dom(f) = \{\emptyset\}$ therefore f is completely determined by the image $f(\emptyset)$ of the only element $\emptyset$ in A^0; for this reason we can identify f by $f(\emptyset)$ and therefore an operation of arity zero on A can be identified with an element of the set A.

We assume that Σ is a set of operation symbols and let $a : \Sigma \longrightarrow N$ be a mapping, that is $dom(a) = \Sigma$. For every $\sigma \in \Sigma$ the natural number $a(\sigma)$ is the arity of σ.

Definition 1.2.2. (*Burmeister (2002)*)
By a **partial Σ-algebra** *we understand a pair*

$$\mathcal{A} = (\mathcal{A}, \{\sigma_\mathcal{A}\}_{\sigma \in \Sigma})$$

where A is a set and for every $\sigma \in \Sigma$ the elemant σ_A is a partial operation on A of arity $a(\sigma)$. In the case when σ_A is an operation of arity $a(\sigma)$ for every $\sigma \in \Sigma$, we say that $\mathcal{A}$ is a **Σ-algebra**. *The system $(a(\sigma))_{\sigma\in\Sigma}$ is the* **signature** *of Σ. For the particular case when Σ is a singleton, i.e. $\Sigma = \{\sigma\}$ for some symbol σ of arity $a(\sigma)$, we say that $\mathcal{A}$ is a* **partial σ-algebra**.

For some values of arity the corresponding operation has a special name. The most encountered cases are the following:

- for $a(\sigma) = 0$ we have a *nullary* operation σ;
- for $a(\sigma) = 1$ we have an *unary* operation;
- for $a(\sigma) = 2$ we have a *binary* operation.

For example, we consider $\Sigma = \{\sigma\}$ and σ a symbol of arity 2. The pair $\mathcal{A} = (A, \{\sigma_A\})$, where $A = \{0, 2, 4\}$, $dom(\sigma_A) = \{(0, 0), (2, 0), (4, 0), (4, 2)\}$ and $\sigma_A(x, y) = x - y$ is a partial Σ-algebra. If we consider $\mathcal{B} = (B, \{\sigma_B\})$, where $B = \{0, 1, 2\}$ and $\sigma_B(x, y) = max\{x, y\}$ then we obtain an example of Σ-algebra.

We consider an equivalence relation ρ on A, where A is the support set of the partial Σ-algebra $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$. The set of all equivalence classes is denoted by $A/_\rho$ and this set is named the *factor set*. Frequently a *congruence* is used instead of an equivalence relation. In comparison with an equivalence relation, a congruence introduces a compatibility property with the operations from $\mathcal{A}$. More precisely, an equivalence relation ρ is a **congruence** if for every $\sigma \in \Sigma$ and every $(x_1, \ldots, x_{a(\sigma)}) \in A^{a(\sigma)}$, $(y_1, \ldots, y_{a(\sigma)}) \in A^{a(\sigma)}$ the following conditions are satisfied: if $(x_1, y_1) \in \rho, \ldots, (x_{a(\sigma)}, y_{a(\sigma)}) \in \rho$ and $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)$ then $(y_1, \ldots, y_{a(\sigma)}) \in dom(\sigma_A)$ and $(\sigma_A(x_1, \ldots, x_{a(\sigma)}), \sigma_A(y_1, \ldots, y_{a(\sigma)})) \in \rho$.

If ρ is a congruence on A then we can obtain the *quotient algebra* denoted by $\mathcal{A}/\rho = (A/\rho, \{\widetilde{\sigma}_A\}_{\sigma\in\Sigma})$, where

$$dom(\widetilde{\sigma}_A) = \{([x_1]_\rho, \ldots, [x_{a(\sigma)}]_\rho) \mid (x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)\}$$
$$\widetilde{\sigma}_A([x_1]_\rho, \ldots, [x_{a(\sigma)}]_\rho) = [\sigma_A(x_1, \ldots, x_{a(\sigma)})]_\rho$$

We observe that this definition does not depend on representatives. Really, if $(x_1, y_1) \in \rho, \ldots, (x_{a(\sigma)}, y_{a(\sigma)}) \in \rho$ and $([x_1]_\rho, \ldots, [x_{a(\sigma)}]_\rho) \in dom(\widetilde{\sigma}_A)$ then $([y_1]_\rho, \ldots, [y_{a(\sigma)}]_\rho) \in dom(\widetilde{\sigma}_A)$ and $\widetilde{\sigma}_A([x_1]_\rho, \ldots, [x_{a(\sigma)}]_\rho) = \widetilde{\sigma}_A([y_1]_\rho, \ldots, [y_{a(\sigma)}]_\rho)$. We observe that the quotient algebra has the same signature as the initial algebra.

In the final part of this section we specify several examples of algebras and we will observe that for each case the operations satisfy certain identities.

Example 1.2.1. (Semigroup, Burris (1981))
A *semigroup* is an algebra $(S, \{\cdot\})$ of signature (2) such that the following property is satisfied:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for each $x, y, z \in S$. If this identity is satisfied then we say that the operation is *associative*.

Example 1.2.2. (Monoid, Burris (1981))
A *monoid* is an algebra $(M, \{\cdot, e\})$ of signature $(2, 0)$ such that the following property is satisfied:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

$$x \cdot 1 = 1 \cdot x = x$$

for each $x, y, z \in M$.

Example 1.2.3. (Group, Burris (1981))
A *group* is an algebra $(G, \{\cdot, ^{-1}, e\})$ of signature $(2, 1, 0)$ such that the following relations are satisfied:

$x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for each $x, y, z \in G$;
$x \cdot e = e \cdot x = x$ for every $x \in G$;
$x \cdot x^{-1} = x^{-1} \cdot x = e$ for every $x \in G$

A group is called a *commutative group* or *Abelian group* if $x \cdot y = y \cdot x$ for every x and y.

Example 1.2.4. (Ring, Burris (1981))
A *ring* is an algebra $(R, \{+, \cdot, -, e\})$ of signature $(2, 2, 1, 0)$ such that the following relations are satisfied:

$(R, +, -, e)$ is an Abelian group;
$(R, \cdot)$ is a semigroup;
$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for each $x, y, z \in R$;
$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ for every $x, y, z \in R$.

A ring with unit is a ring containing an element denoted by 1 such that $x \cdot 1 = 1 \cdot x = x$ for every x.

Other classical algebraic structures such as module, semilattice, lattice, Boolean algebra, Heyting algebra, cylindric algebras can be defined as Σ-algebras (Burris (1981)).

Example 1.2.5. (Partial algebras of binary relations) We consider a nonempty set S. If $\rho_1 \in 2^{S \times S}$ and $\rho_2 \in 2^{S \times S}$ then we define:

$$\rho_1 \circ \rho_2 = \{(x, y) \in S \times S \mid \exists z \in S : (x, z) \in \rho_1, (z, y) \in \rho_2\}$$

We remark that the following case can be encountered: $\rho_1 \neq \emptyset$, $\rho_2 \neq \emptyset$ and nevertheless $\rho_1 \circ \rho_2 = \emptyset$. For some applications the empty relation is not a useful one. In order to avoid this situation we introduce the mapping $prod_S : dom(prod_S) \longrightarrow 2^{S \times S}$ as follows:

$dom(prod_S) = \{(\rho_1, \rho_2) \in 2^{S \times S} \times 2^{S \times S} \mid \rho_1 \circ \rho_2 \neq \emptyset\}$
$prod_S(\rho_1, \rho_2) = \rho_1 \circ \rho_2$

We denote by $R(prod_S)$ the set of all restrictions of the mapping $prod_S$:

$$R(prod_S) = \{u \mid u \prec prod_S\}$$

We observe that if u is an element of $R(prod_S)$ then the pair $(2^{S\times S}, u)$ is a partial algebra. This is a useful partial algebra in the domain of knowledge representation by algebraic methods.

We remark now that although $prod_S$ is an associative operation and $u \prec prod_S$ the operation u can be a non-associative one. Take for example $S = \{x_1, x_2, x_3, x_4, x_5\}$ and the following elements of $2^{S\times S}$:

$\rho_1 = \{(x_1, x_2)\}$; $\rho_2 = \{(x_2, x_3), (x_2, x_4)\}$;
$\rho_3 = \{(x_3, x_4)\}$; $\rho_4 = \{(x_4, x_5)\}$; $\rho_5 = \{(x_2, x_4)\}$;
$\rho_6 = \{(x_2, x_5)\}$; $\rho_7 = \{(x_1, x_5)\}$

Let us consider the mapping $u \prec prod_S$, which is defined as follows:

$$dom(u) = \{(\rho_2, \rho_3), (\rho_5, \rho_4), (\rho_1, \rho_6)\}$$

$$u(\rho_2, \rho_3) = \rho_5; u(\rho_5, \rho_4) = \rho_6; u(\rho_1, \rho_6) = \rho_7$$

We observe that u is not an associative operation. Really, $u(\rho_2, u(\rho_3, \rho_4))$ is not defined, whereas $u(u(\rho_2, \rho_3), \rho_4)) = \rho_6$.

1.3 Closed sets and partial subalgebras

In this section we treat a basic concept of algebra, which is encountered in all classical algebraic structures under the name of *substructure* such as subgroup, sublattice, subspace of a linear space and so an. In general, a substructure of a structure is a subset which is itself a structure if we consider the same operations as in the initial structure. The next definition states this concept in the context of partial algebras.

Definition 1.3.1. *Let $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ be a partial Σ-algebra. A subset $B \subseteq A$ is a* **closed set** *in $\mathcal{A}$ if for every $\sigma \in \Sigma$ the following condition is fulfilled: for every $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A) \cap B^{a(\sigma)}$ we have $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in B$.*

Remark 1.3.1. The empty set is a closed set.

Example 1.3.1. We consider the monoid $\mathcal{N} = (N, \{\sigma_N, \tau_N\})$, which is a Σ-algebra of signature $(2, 0)$, $\Sigma = \{\sigma, \tau\}$ and $\sigma_N : N \times N \longrightarrow N$ is the operation $\sigma_N(x, y) = x + y$ and $\tau_N = 0$. The pair $\mathcal{B} = (B, \{\sigma_B, \tau_B\})$, where $B = \{0, 2, 4, \ldots\}$ is the set of even natural numbers and σ_B, τ_B are the restrictions of σ_N and τ_N respectively, defines the submonoid $\mathcal{B}$ of $\mathcal{N}$.

Definition 1.3.2. *Let $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ be a partial Σ-algebra. If $B \subseteq A$ then* **the closure of** *B* **in** *A is the least closed set containing B and this set is denoted by $\overline{B}$.*

Remark 1.3.2. If B is a closed set the $\overline{B} = B$.

The existence of the closure of a set is obtained immediately from the next proposition.

Proposition 1.3.1. *Let $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ be a partial Σ-algebra and a family $\{X_i\}_{i\in I}$ such that $X_i \subseteq A$ for every $i \in I$. If X_i is a closed set in $\mathcal{A}$ for every $i \in I$ then $\bigcap_{i\in I} X_i$ is a closed set.*

Proof. Take $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A) \cap (\bigcap_{i\in I} X_i)^{a(\sigma)}$. Because X_i is a closed set of $\mathcal{A}$ we have $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in X_i$ and this property is true for every $i \in I$. It follows that $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in \bigcap_{i\in I} X_i$, therefore $\bigcap_{i\in I} X_i$ is a closed set. ■

Corollary 1.3.1. *Let $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ be a partial Σ-algebra and $B \subseteq A$. There is the least closed set $\overline{B}$ which includes B.*

Proof. Take the family $\mathcal{J}$ of all closed sets such that each set includes B. Applying Proposition 1.3.1 we obtain the closed set $\bigcap_{X\in\mathcal{J}} X$ and this is the least closed set which contains B. ■

The following useful proposition is obtained immediately:

Proposition 1.3.2. *The following properties are verified:*

- $\overline{\emptyset} = \emptyset$;
- *If* $X \subseteq Y$ *then* $\overline{X} \subseteq \overline{Y}$;
- $\overline{\overline{X}} = \overline{X}$.

Proof. If $X \subseteq Y$ then the least closed set $\overline{Y}$ which contains Y is a closed set which contains X. But $\overline{X}$ is the least closed set which contains X therefore $\overline{X} \subseteq \overline{Y}$. The set $\overline{X}$ is a closed set therefore by Remark 1.3.2 we have $\overline{\overline{X}} = \overline{X}$. ■

Remark 1.3.3. The property $\overline{X \cap Y} = \overline{X} \cap \overline{Y}$ is not true. Really, let us consider the Σ-algebra $\mathcal{A} = (A, \{\sigma_A\})$ of signature (2), where $A = N$ and $\sigma_A(x, y) = x + y$. Take $X = \{1, 4\}$ and $Y = \{2, 4\}$. We have $X \cap Y = \{4\}$, therefore $\overline{X \cap Y} = \{4n\}_{n\geq 1}$. On the other hand we have $\overline{X} = N \setminus \{0\}$, $\overline{Y} = \{2n\}_{n\geq 1}$. Thus we have $2 \in \overline{X} \cap \overline{Y}$, but $2 \notin \overline{X \cap Y}$.

A concept which is frequently encountered in the theory of universal algebras is given in the next definition.

Definition 1.3.3. (*Burris (1981)*)
A mapping $C : 2^A \longrightarrow 2^A$ is called a **closure operator** *on A if for every $X, Y \in 2^A$ the following properties are satisfied:*

- (*extensive*) $X \subseteq C(X)$;
- (*idempotent*) $C(C(X)) = C(X)$;
- (*monotone*) $X \subseteq Y \Longrightarrow C(X) \subseteq C(Y)$.

The following property is obtained immediately:

Proposition 1.3.3. *Let $\mathcal{A} = (A, \{\sigma_A\}_{\sigma \in \Sigma})$ be a partial Σ-algebra. The operator $Cl : 2^A \longrightarrow 2^A$ defined by $Cl(X) = \overline{X}$, where $\overline{X}$ is the least closed set containing X, is a closure operator.*

Proof. Immediate by Proposition 1.3.2. ■

In what follows we are interested to give an algorithm to compute the closure $\overline{B}$ of the set B. The next two propositions give such algorithms.

Proposition 1.3.4. *Let be $B \subseteq A$, $B \neq \emptyset$. If we consider the sequence*

$$\begin{cases} B_0 = B \\ B_{n+1} = B_n \cup \{\ \sigma_A(x) \mid \sigma \in \Sigma, x \in dom(\sigma_A) \cap B_n^{a(\sigma)}\}, \quad n \geq 0 \end{cases} \tag{1.3}$$

then $\{B_n\}_{n \geq 0}$ is an increasing sequence and $\overline{B} = \bigcup_{n \geq 0} B_n$.

Proof. We denote $C = \bigcup_{n \geq 0} B_n$. From (1.3) we have $B_n \subseteq B_{n+1}$ for every n. Let us verify that C is a closed set. Take $(x_1, \ldots, x_{a(\sigma)}) \in C^{a(\sigma)} \cap dom(\sigma_A)$. There are $n_1, \ldots, n_{a(\sigma)} \in N$ such that $x_j \in B_{n_j}$ for $j = 1, \ldots, a(\sigma)$. Because $\{B_n\}_{n \geq 0}$ is an increasing sequence we have $(x_1, \ldots, x_{a(\sigma)}) \in B_k^{a(\sigma)}$, where $k = max\{n_1, \ldots, n_{a(\sigma)}\}$. From (1.3) we deduce that $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in B_{k+1}$, therefore $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in C$. In order to show that C is the least closed set which contains B we consider an arbitrary set Z such that

$Z \supseteq B$
Z is a closed set

and we prove that $C \subseteq Z$. We prove by induction on $i \geq 0$ that

$$B_i \subseteq Z \tag{1.4}$$

For $i = 0$ the inclusion (1.4) is true because $B_0 = B$ and $B \subseteq Z$. Suppose (1.4) is true for $i = m$ and take $x \in B_{m+1}$. Two cases are possible:

- $x \in B_m$.
  By the inductive assumption we have $x \in Z$.
- $x \in B_{m+1} \setminus B_m$.
  There are $\sigma \in \Sigma$ and $(x_1, \ldots, x_{a(\sigma)}) \in B_m^{a(\sigma)} \cap dom(\sigma_A)$ such that $x = \sigma_A(x_1, \ldots, x_{a(\sigma)})$. By the inductive assumption we have $B_m \subseteq Z$. On the other hand Z is a closed set, therefore $x \in Z$.

Now the proposition is proved because from (1.4) we have $\bigcup_{n \geq 0} B_n \subseteq Z$. ■

An useful property is stated in the following proposition:

Proposition 1.3.5. *Let be $B \subseteq A$, $B \neq \emptyset$. If we consider the sequence*

$$\begin{cases} C_0 = B \\ C_{n+1} = B \cup \{\ \sigma_A(x) \mid \sigma \in \Sigma, x \in dom(\sigma_A) \cap C_n^{a(\sigma)}\}, \quad n \geq 0 \end{cases} \tag{1.5}$$

then $\{C_n\}_{n \geq 0}$ is an increasing sequence and $\overline{B} = \bigcup_{n \geq 0} C_n$.

Proof. It is easy to verify by induction on n that $C_n = B_n$ for every $n \geq 0$, where the sequence $\{B_n\}_{n\geq 0}$ is defined in (1.3).
Obviously we have $C_0 = B_0$. We suppose $C_n = B_n$ and we prove that $C_{n+1} = B_{n+1}$. Take an element $x \in C_{n+1}$. The following two cases are possible:

- $x \in B$
  In this case $x \in B_{n+1}$ because $B = B_0 \subseteq B_{n+1}$.
- $x = \sigma_A(x_1, \ldots, x_{a(\sigma)})$ for some $\sigma \in \Sigma$ and $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A) \cap C_n^{a(\sigma)}$. By the inductive assumption we have $C_n = B_n$ and therefore $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A) \cap B_n^{a(\sigma)}$. From (1.3) we obtain $x \in B_{n+1}$.

Now we take $x \in B_{n+1}$. By similarity with the previous analysis we have also two cases:

- $x \in B$
  In this case $x \in C_{n+1}$ because $B \subseteq C_{n+1}$.
- $x = \sigma_A(x_1, \ldots, x_{a(\sigma)})$ for some $\sigma \in \Sigma$ and $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A) \cap B_n^{a(\sigma)}$. By the inductive assumption we have $B_n = C_n$ and therefore $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A) \cap C_n^{a(\sigma)}$. From (1.5) we obtain $x \in C_{n+1}$.

By Proposition 1.3.4 we have $\overline{B} = \bigcup_{n\geq 0} B_n$, therefore $\overline{B} = \bigcup_{n\geq 0} C_n$ because $B_n = C_n$ for every $n \geq 0$ and the proposition is proved. ∎

Definition 1.3.4. *Let $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ be a partial Σ-algebra. If $B \subseteq A$ and $\overline{B} = A$ then we say that the partial Σ-algebra $\mathcal{A}$ is* **generated** *by B.*

Definition 1.3.5. *Let us consider a partial Σ-algebra $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$. A* **partial subalgebra** *of $\mathcal{A}$ is a pair $\mathcal{B} = (B, \{\sigma_B\}_{\sigma\in\Sigma})$, where*

- *$B \subseteq A$ is a closed set;*
- *for each $\sigma \in \Sigma$ we have $dom(\sigma_B) = B^{a(\sigma)} \cap dom(\sigma_A)$ and*

$$\sigma_B(x_1, \ldots, x_{a(\sigma)}) = \sigma_A(x_1, \ldots, x_{a(\sigma)})$$

  for every $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_B)$

If $C \subseteq A$ is a nonempty set and take $B = \overline{C}$ the we obtain the partial subalgebra of $\mathcal{A}$ **generated by** *C.*

We remark that a partial subalgebra of a partial Σ-algebra is itself a partial Σ-algebra with respect to the restrictions of the operations σ_A to the elements of B.

The next proposition states a result which gives a method by which we prove some property for a given set of elements. We denote by $\mathcal{P}$ some property. If an element x has the property $\mathcal{P}$ then we denote this fact by $\mathcal{P}(x)$.

Proposition 1.3.6. (*algebraic induction*)
Let $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ be a partial Σ-algebra, a subset $B \subseteq A$ and a property $\mathcal{P}$. If

- (*initial step*) *for every* $x \in B$ *we have* $\mathcal{P}(x)$*;*
- (*inductive step*) *for every* $\sigma \in \Sigma$ *and every* $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)$ *from* $P(x_i)$ *for* $i = 1, \ldots, a(\sigma)$ *we deduce* $P(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$

then for every $x \in \overline{B}$ *we have* $\mathcal{P}(x)$.

Proof. We denote $D = \{x \in A \mid \mathcal{P}(x)\}$. From the initial step we have $B \subseteq D$ and from the inductive step we deduce that D is a closed set. But $\overline{B}$ is the least subset of A which is a closed set and includes B, therefore $\overline{B} \subseteq D$.
■

Remark 1.3.4. The algebraic principle is known in literature also as **structural induction**. Various books (Rudeanu (1991), Burmeister (2002)) treat this fundamental concept.

1.4 Morphisms of partial algebras

The concept of homomorphism is a central concept in algebra. A lot of properties are based on this concept. For the general case of partial algebras this concept is defined as follows.

Definition 1.4.1. *We consider the partial* Σ*-algebras* $\mathcal{A}=(A, \{\sigma_A\}_{\sigma\in\Sigma})$ *and* $\mathcal{B}=(B, \{\sigma_B\}_{\sigma\in\Sigma})$. *We say that the mapping* $h : A \longrightarrow B$ *is a* **homomorphism** *or a* **morphism** *of partial algebras from* $\mathcal{A}$ *to* $\mathcal{B}$ *if for every* $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)$ *the following conditions are fulfilled:*

- $(h(x_1), \ldots, h(x_{a(\sigma)})) \in dom(\sigma_B)$
- $\sigma_B(h(x_1), \ldots, h(x_{a(\sigma)})) = h(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$

A bijective homomorphism is an **isomorphism**.

In order to give an intuitive representation of this concept we define the *product mapping*

$$h^{a(\sigma)} : A^{a(\sigma)} \longrightarrow B^{a(\sigma)}$$

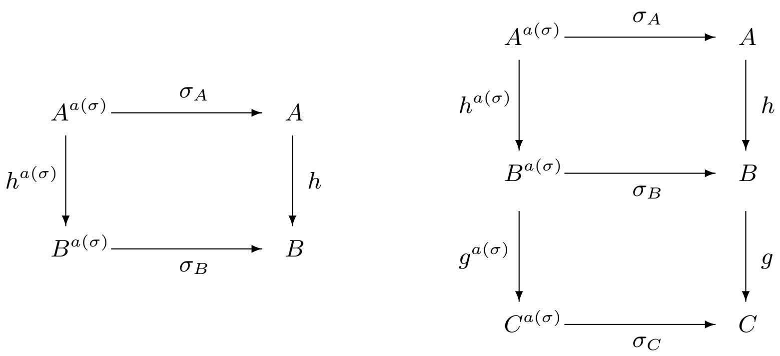
by $h^{a(\sigma)}(x_1, \ldots, x_{a(\sigma)}) = (h(x_1), \ldots, h(x_{a(\sigma)}))$.

Because h is a homomorphism of partial algebras, in the diagram of Figure 1.2a) we have the following property: if we are able to go along the path $(A^{a(\sigma)}, A, B)$ then we are able also to go along the path $(A^{a(\sigma)}, B^{a(\sigma)}, B)$ and we obtain the same result.

Proposition 1.4.1. (*Rasiowa and Sikorski (1963)*) *We consider the partial* Σ*-algebras* $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$, $\mathcal{B} = (B, \{\sigma_B\}_{\sigma\in\Sigma})$ *and* $\mathcal{C} = (C, \{\sigma_C\}_{\sigma\in\Sigma})$. *If* $h : A \longrightarrow B$ *and* $g : B \longrightarrow C$ *are morphisms then the superposition* $g \circ h$ *is a morphism from* $\mathcal{A}$ *to* $\mathcal{C}$.

Proof.

The morphisms h and g are represented in Figure 1.2b). Because h is a morphism the following conditions are fulfilled for every $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)$:



a) b)

Fig. 1.2. Morphism diagrams

- $(h(x_1), \ldots, h(x_{a(\sigma)}) \in dom(\sigma_B)$
- $\sigma_B(h(x_1), \ldots, h(x_{a(\sigma)})) = h(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$

But g is also a morphism of partial algebras, therefore

- $(g(h(x_1)), \ldots, g(h(x_{a(\sigma)})) \in dom(\sigma_C)$
- $\sigma_C(g(h(x_1)), \ldots, g(h(x_{a(\sigma)}))) = g(\sigma_B(h(x_1), \ldots, h(x_{a(\sigma)})))$

It follows that for every $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)$ we have

- $(g \circ h(x_1), \ldots, g \circ h(x_{a(\sigma)}) \in dom(\sigma_C)$
- $\sigma_C(g \circ h(x_1), \ldots, g \circ h(x_{a(\sigma)})) = g \circ h(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$

because $g(\sigma_B(h(x_1), \ldots, h(x_{a(\sigma)}))) = g(h(\sigma_A(x_1, \ldots, x_{a(\sigma)})))$. ■

The properties from the next proposition allow us to identify several properties on particular cases such that an economy of proofs are obtained.

Proposition 1.4.2. (*Căzănescu (1975)*)
Let us consider the partial Σ-algebras $\mathcal{A}=(A, \{\sigma_A\}_{\sigma\in\Sigma})$ and $\mathcal{B}=(B, \{\sigma_B\}_{\sigma\in\Sigma})$ and a morphism $h : A \longrightarrow B$.

1) *If $Y \subseteq B$ is a closed set in $\mathcal{B}$ then $h^{-1}(Y)$ is a closed set in $\mathcal{A}$.*
2) *If $\mathcal{A}$ is an algebra and $X \subseteq A$ is a closed set in $\mathcal{A}$ then $h(X)$ is a closed set in $\mathcal{B}$.*
3) *If $X \subseteq A$ then $h(\overline{X}) \subseteq \overline{h(X)}$. If $\mathcal{A}$ is an algebra then $h(\overline{X}) = \overline{h(X)}$.*

Proof.
1) Consider a closed set $Y \subseteq B$. Take an arbitrary element $\sigma \in \Sigma$ and $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)$ such that $x_1, \ldots, x_{a(\sigma)} \in h^{-1}(Y)$. Because h is a morphism we have $(h(x_1), \ldots, h(x_{a(\sigma)})) \in dom(\sigma_B)$ and

$$\sigma_B(h(x_1), \ldots, h(x_{a(\sigma)})) = h(\sigma_A(x_1, \ldots, x_{a(\sigma)})) \tag{1.6}$$

On the other hand Y is a closed set in $\mathcal{B}$, therefore $\sigma_B(h(x_1),\ldots,h(x_{a(\sigma)}))\in B$ because $h(x_1),\ \ldots,h(x_{a(\sigma)})\in Y$ and $(h(x_1),\ \ldots,h(x_{a(\sigma)}))\in dom(\sigma_B)$. From (1.6) we obtain $h(\sigma_A(x_1,\ldots,x_{a(\sigma)}))\in Y$ therefore $\sigma_A(x_1,\ldots,x_{a(\sigma)})\in h^{-1}(Y)$ and thus the first property is proved.

2) In order to verify the second property we take $\sigma\in\Sigma$ and $y_1,\ldots,y_{a(\sigma)}\in h(X)$. There are $x_1,\ldots,x_{a(\sigma)}\in X$ such that $y_1=h(x_1),\ldots,y_{a(\sigma)}=h(x_{a(\sigma)})$. We have $dom(\sigma_A)=A^{a(\sigma)}$ because $\mathcal{A}$ is an algebra. The mapping h is a morphism therefore $(h(x_1),\ldots,h(x_{a(\sigma)}))\in dom(\sigma_B)$ and

$$\sigma_B(h(x_1),\ldots,h(x_{a(\sigma)}))=h(\sigma_A(x_1,\ldots,x_{a(\sigma)})) \qquad (1.7)$$

But X is a closed set in $\mathcal{A}$, therefore $\sigma_A(x_1,\ldots,x_{a(\sigma)}))\in X$ and thus $h(\sigma_A(x_1,\ldots,x_{a(\sigma)})))\in h(X)$. Taking into account the fact that $h(x_i)=y_i$, from (1.7) we obtain $\sigma_B(y_1,\ldots,y_{a(\sigma)})\in h(X)$.

3) In order to prove the last property we observe that based on (1.1) and (1.2) we have

$$X\subseteq h^{-1}(h(X))\subseteq h^{-1}(\overline{h(X)}) \qquad (1.8)$$

By the first property already proved, the set $h^{-1}(\overline{h(X)})$ is a closed set in $\mathcal{A}$ because $\overline{h(X)}$ is a closed set in $\mathcal{B}$. From (1.8) and the fact that $\overline{X}$ is the least closed set which contains X we obtain $\overline{X}\subseteq h^{-1}(\overline{h(X)})$ therefore $h(\overline{X})\subseteq\overline{h(X)}$. If $\mathcal{A}$ is an algebra then $h(\overline{X})$ is a closed set in $\mathcal{B}$. But $h(\overline{X})\supseteq h(X)$, therefore $h(\overline{X})\supseteq\overline{h(X)}$ because $\overline{h(X)}$ is the least closed set which contains $h(X)$. ■

The concept introduced in the next definition as well as the concept of morphisms of partial algebras are useful to study a method of knowledge representation based on stratified graphs (Ţăndăreanu (2000a)).

Definition 1.4.2. *Let* $\mathcal{A}=(A,\{\sigma_A\}_{\sigma\in\Sigma})$ *be a partial* Σ*-algebra. The set* $B\subseteq A$ *is an* **initial set** *of* $\mathcal{A}$ *if for every* $\sigma\in\Sigma$ *and every* $(x_1,\ldots,x_{a(\sigma)})\in dom(\sigma_A)$ *the following condition is satisfied:*

$$\sigma_A(x_1,\ldots,x_{a(\sigma)})\in B\Longrightarrow\{x_1,\ldots,x_{a(\sigma)}\}\subseteq B$$

If this condition is satisfied then we write $B\in Initial(\mathcal{A})$.

Example 1.4.1. Consider the σ-algebra $\mathcal{N}_1=(N,\sigma_N)$ of signature (2) and $\sigma_N(x,y)=x+y$. Take $B=\{2,4,8\}\subseteq N$. We have $\sigma_N(2,6)=8\in B$ but $6\notin N$. Thus $B\notin Initial(\mathcal{N}_1)$.

Example 1.4.2. Let us consider σ-algebra $\mathcal{N}_2=(N,\sigma_N)$ of signature (1) and $\sigma_N(x)=x+2$. Take $B=\{2k\}_{k\geq 0}\subseteq N$. If $\sigma_N(m)\in B$ then $m+2$ is an even number therefore $m\in B$. Thus $B\in Initial(\mathcal{N}_2)$.

Proposition 1.4.3. (*Căzănescu (1975)*) *If* $B_i\in Initial(\mathcal{A})$ *for each* $i\in I$ *then* $\bigcup_{i\in I}B_i\in Initial(\mathcal{A})$ *and* $\bigcap_{i\in I}B_i\in Initial(\mathcal{A})$.

Proof. Consider an arbitrary element $\sigma \in \Sigma$ and $(x_1, \ldots, x_{a(\sigma)}) \in dom(\sigma_A)$ such that $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in \bigcup_{i \in I} B_i$. There is $k \in I$ such that $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in B_k$. But $B_k \in Initial(\mathcal{A})$ therefore $x_1, \ldots, x_{a(\sigma)} \in B_k$. It follows that $x_1, \ldots, x_{a(\sigma)} \in \bigcup_{i \in I} B_i$. A similar proof can be given for intersection. ■

Finally, we can define a concept which is used in the next chapter.

Definition 1.4.3. *The Σ-algebra $\mathcal{A}=(A, \{\sigma_A\}_{\sigma \in \Sigma})$ is* **free generated** *by $M \subseteq A$ if for every Σ-algebra $\mathcal{B}=(B, \{\sigma_B\}_{\sigma \in \Sigma})$ and every mapping $f : M \longrightarrow B$ there exists a morphism $h : A \longrightarrow B$, uniquely determined, such that $f \prec h$.*

An expressive graphical representation of this condition is given in Figure 1.3, where i denotes the inclusion mapping: $i(x) = x$ for every $x \in M$. The condition can be stated also as follows: every mapping $f : M \longrightarrow B$ can be uniquely extended to a morphism $h : A \longrightarrow B$ of Σ-algebras.

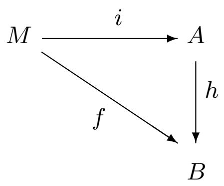


Fig. 1.3. The extension of f

Example 1.4.3. Consider the σ-algebra $\mathcal{N} = (N, \sigma_N)$, where $\sigma_N(x) = x + 1$. Take $M = \{0\}$. We verify now the following two properties:

- $\mathcal{N}$ is generated by M
- $\mathcal{N}$ is free generated by M

Applying Proposition 1.3.4 we obtain:

$$\begin{cases} M_0 = \{0\} \\ M_n = \{0, 1, \ldots, n\} \end{cases} \tag{1.9}$$

We verify by induction (1.9). For $n = 0$ this relation is true. Suppose (1.9) is true for n. Based on (1.3) we obtain

$$M_{n+1} = M_n \cup \{n + 1\} = \{0, 1, \ldots, n + 1\}$$

therefore (1.9) is true for $n + 1$.
It follows that

$$\overline{M} = \bigcup_{n \geq 0} M_n = N$$

therefore the closure of M is N.

Let us consider the graphical representation from Figure 1.3, where $A = N$ and $\mathcal{B} = (B, \sigma_B)$ is a σ-algebra of signature (1). Consider the mapping $h : N \longrightarrow B$ defined as follows:

$$\begin{cases} h(0) = f(0) \\ h(1) = \sigma_B(f(0)) \\ h(2) = \sigma_B(\sigma_B(f(0))) \\ \dots\dots\dots\dots \\ h(n+1) = \sigma_B^{n+1}(f(0)) \end{cases}$$

The mapping h is a morphism of σ-algebras. Really, $\sigma_B(h(n)) = \sigma_B^{n+1}(f(0))$ and $h(\sigma_N(n)) = h(n+1) = \sigma_B^{n+1}(f(0))$ therefore $\sigma_B(h(n)) = h(\sigma_N(n))$. Thus the mapping f can be extended to a morphism of σ-algebras.

Suppose $g : N \longrightarrow B$ is a morphism of σ-algebras such that $f \prec g$. It follows that $g(0) = f(0) = h(0)$. We suppose $g(n) = h(n)$ and we prove that $g(n+1) = h(n+1)$. Let us use the fact that g is a morphism. We obtain

$$g(n+1) = g(\sigma_N(n)) = \sigma_B(g(n)) = \sigma_B(h(n)) = h(\sigma_N(n)) = h(n+1)$$

Thus $g = h$ and N is free generated by $\{0\}$. This example is used also in the next chapter.

2. Peano algebras

In this chapter we introduce the concept of Peano algebra, we show that every Peano algebra is a free generated algebra, we prove the existence of the Peano algebras and we give a method to build such structures.

2.1 Definitions and intermediate results

We begin this section by defining the concept of Peano algebra over some set. In the remainder of this section we establish the prerequisites for the next sections of this chapter.

Definition 2.1.1. *A Σ-algebra $\mathcal{A}=(A, \{\sigma_A\}_{\sigma\in\Sigma})$ is a* **Peano Σ-algebra** *over $M \subseteq A$ if the following conditions are satisfied:*

1) $\overline{M} = A$;
2) $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \notin M$ *for every* $\sigma \in \Sigma$ *and every* $x_1, \ldots, x_{a(\sigma)} \in A$;
3) *for every* $\sigma, \tau \in \Sigma$ *and every* $x_1, \ldots, x_{a(\sigma)} \in A$, $y_1, \ldots, y_{a(\tau)} \in A$ *we have*

$$\sigma_A(x_1, \ldots, x_{a(\sigma)}) = \tau_A(y_1, \ldots, y_{a(\tau)}) \Longrightarrow \sigma = \tau, x_i = y_i, i = 1, \ldots, a(\sigma)$$

If $\Sigma = \{\sigma\}$ is a singleton then the corresponding structure is called shortly **Peano σ-algebra**.

Directly from this definition we remark the following aspects:

- A Peano algebra is a *total algebra* and not a partial one.
- The support set of a Peano algebra is generated by some of its subsets.
- The elements of the set M generating the support set A are viewed as "atomic" elements. This means that these elements can not be decomposed into some elements by means of the algebraic operations. In other words none element of M can be obtained by composing other elements of the support set.
- Each element of the set $A \setminus M$ is *uniquely* written as $\sigma_A(x_1, \ldots, x_{a(\sigma)})$.

Remark 2.1.1. Let us consider again the σ-algebra $\mathcal{N} = (N, \sigma_N)$, where $\sigma_N(x) = x + 1$, taken in Example 1.4.3. Let us verify that $\mathcal{N}$ is a Peano σ-algebra over $M = \{0\}$.

1) The condition $\overline{M} = N$ is true as it was shown in Example 1.4.3.

2) For every $n \in N$ we have $\sigma_N(n) \notin M$ because $n + 1 \geq 1$.

3) For every $n, k \in N$ if $\sigma_N(n) = \sigma_N(k)$ then $n + 1 = k + 1$ therefore $n = k$.

If we choose $M = \{0, 1\}$ then the first condition is verified as well as the third condition. But $1 \in M$ and $1 = \sigma_N(0)$, therefore the second condition is not verified.

Remark 2.1.2. An interesting aspect connected by the concept of Peano algebra is discussed in Burris (1981). We remember he following axioms, known as Peano's axioms for natural numbers:

1) 0 is a natural number.
2) Every natural number a has a successor, denoted by Sa.
3) Distinct natural numbers have distinct successors: $n = k$ if and only if $Sn = Sk$.
4) No natural number has 0 as its successor.
5) We denote by Q a property for natural numbers. If $Q(0)$ is true and from $Q(n)$ we can prove $Q(n+1)$ then Q will hold for all natural numbers.

It is not difficult to observe that all these properties are encountered in the context of Peano algebras and this fact can explain the name of the structure introduced in Definition 2.1.1. As a matter of fact, we can immediate identify the property $\overline{M} = N$ for $M = \{0\}$ and the extension of the Peano's axiom to the conditions from Definition 2.1.1. Moreover, the last axiom of Peano postulates the proof method known as mathematical induction (induction over the naturals). This method is encountered in the theory of universal algebras as algebraic induction.

The property stated in the next lemma gives a basic result used to prove a fundamental property of the Peano Σ-algebras.

Lemma 2.1.1. *Let $\mathcal{A}$=($A, \{\sigma_A\}_{\sigma \in \Sigma}$) be a Peano Σ-algebra over M. We consider a set B and for each $\sigma \in \Sigma$ we take a mapping*

$$f_\sigma : B^{a(\sigma)} \times A^{a(\sigma)} \longrightarrow B$$

For every mapping $f : M \longrightarrow B$ there is a mapping $h : A \longrightarrow B$, uniquely determined, such that

$$f \prec h \tag{2.1}$$

$$h(\sigma_A(x_1, \ldots, x_{a(\sigma)})) = f_\sigma(h(x_1), \ldots, h(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)}) \tag{2.2}$$

for every $\sigma \in \Sigma$ and $x_1, \ldots, x_{a(\sigma)} \in A$.

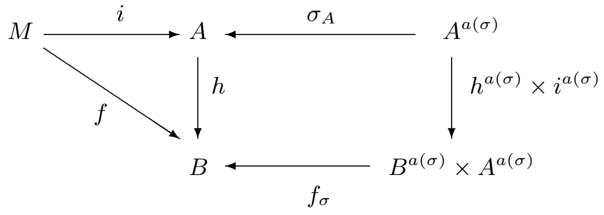


Fig. 2.1. Commutative diagram

Proof. The property stated in this lemma can be restated as we can view in Figure 2.1: every mapping f can be uniquely extended to h such that the "rectangular" diagram is commutative.

Because $\overline{M} = A$, using (1.5) we obtain

$$A = \bigcup_{n \geq 0} M_n \tag{2.3}$$

where $\{M_n\}_{n \geq 0}$ is the increasing sequence given by

$$\begin{cases} M_0 = M \\ M_{n+1} = M \cup \{ \, \sigma_A(x) \mid \sigma \in \Sigma, x \in M_n^{a(\sigma)} \}, \quad n \geq 0 \end{cases} \tag{2.4}$$

For $n \geq 0$ we define inductively the mappings $f_n : M_n \longrightarrow val(f_n)$ as follows:

- $f_0 = f$;
- $f_{n+1}(x) = f(x)$ if $x \in M$
- $f_{n+1}(x) = f_\sigma(f_n(x_1), \ldots, f_n(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)})$ if $x_1, \ldots, x_{a(\sigma)} \in M_n$ and $x = \sigma_A(x_1, \ldots, x_{a(\sigma)})$

For the sequence $\{f_n\}_{n \geq 0}$ we prove the following properties:

1) $\{f_n\}_{n \geq 0}$ *is a well defined sequence*

First we prove that for each natural number $n \geq 0$ we have the following properties:

$$M \cap \{ \, \sigma_A(x) \mid \sigma \in \Sigma, x \in dom(\sigma_A) \cap M_n^{a(\sigma)} \} = \emptyset \tag{2.5}$$

$$val(f_n) \subseteq B \tag{2.6}$$

The relation (2.5) is immediately obtained from Definition 2.1.1. The relation (2.6) can be verified by induction on n. For $n = 0$ this relation is true because $f_0 = f$ and $val(f) \subseteq B$. Suppose $val(f_n) \subseteq B$ and take $y \in val(f_{n+1})$. There exist $\sigma \in \Sigma$ and $x_1, \ldots, x_{a(\sigma)} \in M_n$ such that $y = f_{n+1}(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$. But $f_n(x_1), \ldots, f_n(x_{a(\sigma)}) \in val(f_n)$ and by the inductive assumption $val(f_n) \subseteq B$. It follows that

$$(f_n(x_1), \ldots, f_n(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)}) \in B^{a(\sigma)} \times A^{a(\sigma)}$$

therefore $f_\sigma(f_n(x_1), \ldots, f_n(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)})$ is defined and moreover, it is an element of B. But $y = f_{n+1}(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$ and using the definition of f_{n+1} we obtain $y \in B$.
By Definition 2.1.1 we know that every element $x \in A \setminus M$ is uniquely written as $\sigma_A(x_1, \ldots, x_{a(\sigma)})$ for some $\sigma \in \Sigma$ and $x_1, \ldots, x_{a(\sigma)} \in A$. Taking into account the relations (2.5) and (2.6) we observe that the sequence $\{f_n\}_{n \geq 0}$ is well defined.

2) $f_n \prec f_{n+1}$ *for every* $n \geq 0$

For $n = 0$ this property is true by the definition of f_0 and f_1. Suppose that $f_n \prec f_{n+1}$. Let us prove that $f_{n+1} \prec f_{n+2}$. Take $x \in M_{n+1}$. If $x \in M$ then $f_{n+2}(x) = f(x) = f_{n+1}(x)$. Otherwise, there is $\sigma \in \Sigma$ and there are $x_1, \ldots, x_{a(\sigma)} \in M_n$ such that $x = \sigma_A(x_1, \ldots, x_{a(\sigma)})$. We have $x \in M_{n+1} \subseteq M_{n+2}$ and applying the definition for the sequence $\{f_n\}_{n \geq 0}$ we obtain:

$$f_{n+1}(x) = f_\sigma(f_n(x_1), \ldots, f_n(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)}) \tag{2.7}$$

$$f_{n+2}(x) = f_\sigma(f_{n+1}(x_1), \ldots, f_{n+1}(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)}) \tag{2.8}$$

But $x_1 \in M_n$, …, $x_{a(\sigma)} \in M_n$ and by the inductive assumption we have $f_n \prec f_{n+1}$. It follows that

$$f_{n+1}(x_1) = f_n(x_1)$$

$$\ldots\ldots$$

$$f_{n+1}(x_{a(\sigma)}) = f_n(x_{a(\sigma)})$$

and from (2.7) and (2.8) we obtain $f_{n+2}(x) = f_{n+1}(x)$.

We define now the mapping $h : A \longrightarrow B$ by $h(x) = f_n(x)$ if $x \in M_n$. This is a well defined mapping because we have (2.3) and $f_n \prec f_{n+1}$.
We observe that for every $x \in M = M_0$ we have $h(x) = f_0(x) = f(x)$, therefore we have (2.1).
Let us take an arbitrary element $\sigma \in \Sigma$ and $x_1, \ldots, x_{a(\sigma)} \in A$. From (2.3) and the monotony of the sequence $\{M_n\}_{n \geq 0}$ we deduce that there is $k \in N$ such that $x_1, \ldots, x_{a(\sigma)} \in M_k$. It follows that $\sigma_A(x_1, \ldots, x_{a(\sigma)}) \in M_{k+1}$ and thus

$$h(\sigma_A(x_1, \ldots, x_{a(\sigma)})) = f_{k+1}(\sigma_A(x_1, \ldots, x_{a(\sigma)})) =$$
$$f_\sigma(f_k(x_1), \ldots, f_k(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)}) =$$
$$f_\sigma(h(x_1), \ldots, h(x_{a(\sigma)}), x_1, \ldots, x_{a(\sigma)})$$

and therefore we have (2.2).

Let us prove that h is uniquely determined. We suppose that there is $g : A \longrightarrow B$ such that (2.1) and (2.2) are satisfied. We verify by algebraic induction that $h(x) = g(x)$ for every $x \in A$. We have $h(x) = g(x)$ for every $x \in M$. We take $\sigma \in \Sigma$ and $x_1, \ldots, x_{a(\sigma)} \in A$ such that $h(x_i) = g(x_i)$ for $i = 1, \ldots, a(\sigma)$. Using (2.2) we have

$$g(\sigma_A(x_1,\ldots,x_{a(\sigma)})) = f_\sigma(g(x_1),\ldots,g(x_{a(\sigma)}),x_1,\ldots,x_{a(\sigma)}) =$$

$$f_\sigma(h(x_1),\ldots,h(x_{a(\sigma)}),x_1,\ldots,x_{a(\sigma)}) = h(\sigma_A(x_1,\ldots,x_{a(\sigma)}))$$

and the proposition is proved. ∎

Proposition 2.1.1. *If $\mathcal{A}=(A,\{\sigma_A\}_{\sigma\in\Sigma})$ is a Peano Σ-algebra over M then $\mathcal{A}$ is a Σ-algebra free generated by M.*

Proof. Let $\mathcal{B} = (B,\{\sigma_B\}_{\sigma\in\Sigma})$ be a Σ-algebra and a mapping $f : M \longrightarrow B$. For each $\sigma \in \Sigma$ we define the mapping $f_\sigma : B^{a(\sigma)} \times A^{a(\sigma)} \longrightarrow B$ as follows:

$$f_\sigma(y_1,\ldots,y_{a(\sigma)},x_1,\ldots,x_{a(\sigma)}) = \sigma_B(y_1,\ldots,y_{a(\sigma)})$$

By Lemma 2.1.1 we deduce that f can be uniquely extended to the mapping h such that (2.2) is satisfied. Based on this relation we obtain

$$h(\sigma_A(x_1,\ldots,x_{a(\sigma)})) = f_\sigma(h(x_1),\ldots,h(x_{a(\sigma)}),x_1,\ldots,x_{a(\sigma)}) =$$

$$\sigma_B(h(x_1),\ldots,h(x_{a(\sigma)}))$$

therefore h is a morphism. ∎

2.2 A method to build Peano algebras

The next proposition establishes the existence of the Peano Σ-algebras over some set and based on the results treated in the previous section a method to build such structures is obtained.

Proposition 2.2.1. *For every sets M and Σ such that $\Sigma \cap M = \emptyset$ there exists a Peano Σ-algebra over M.*

Proof. We consider the Σ-algebra $\mathcal{K} = (K,\{\sigma_K\}_{\sigma\in\Sigma})$, where the support set K is the set $(\Sigma \cup M)^+$ of all nonempty words over the alphabet $\Sigma \cup M$ and for each $\sigma \in \Sigma$:

$$\sigma_K(x_1,\ldots,x_{a(\sigma)}) = \sigma x_1 \ldots x_{a(\sigma)}$$

Take $A = \overline{M}$, the closure of M in $\mathcal{K}$ and consider the subalgebra $\mathcal{A} = (A,\{\sigma_A\}_{\sigma\in\Sigma})$ of the Σ-algebra $\mathcal{K}$. Let us prove that $\mathcal{A}$ is a Peano Σ-algebra over M. To do this we have to verify the conditions specified in Definition 2.1.1. The first condition is satisfied by definition of the set A. In order to verify the second condition, we suppose the contrary: there is $\sigma \in \Sigma$ and there are $x_1,\ldots,x_{a(\sigma)} \in A$ such that $\sigma_A(x_1,\ldots,x_{a(\sigma)}) \in M$. It follows that $\sigma x_1 \ldots x_{a(\sigma)} \in M$, therefore $a(\sigma) = 0$ and $\sigma \in M$. This is not possible because $\Sigma \cap M = \emptyset$.
Let us prove that

$$p\alpha = q\beta, p \in M, q \in A, \alpha,\beta \in (\Sigma \cup M)^+ \Longrightarrow q \in M$$

By contrary we suppose that $q \in A \backslash M$. Because A is generated by M we have $q = \sigma_K(z_1, \ldots, z_{a(\sigma)})$ for some $\sigma \in \Sigma$ and $z_1, \ldots, z_{a(\sigma)} \in A$. But $p\alpha = q\beta$, therefore $p\alpha = \sigma z_1 \ldots z_{a(\sigma)}\beta$. It follows that $p = \sigma$, which is not true because $\Sigma \cap M = \emptyset$.

Now we prove by algebraic induction that for each $x \in A$ the following property Q is true:

$$x\alpha = y\beta, y \in A, \alpha, \beta \in (\Sigma \cup M)^+ \Longrightarrow x = y, \alpha = \beta$$

For $x \in M$, as we proved above we have $y \in M$. Thus $x = y$ and $\alpha = \beta$. In order to verify the inductive step we take $\sigma \in \Sigma$ and consider $x_1, \ldots, x_{a(\sigma)}$ such that $Q(x_1)$, ..., $Q(x_{a(\sigma)})$ are true. Let us prove that $Q(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$ is also true. In order to verify this property we suppose that

$$\sigma_A(x_1, \ldots, x_{a(\sigma)})\alpha = y\beta$$

where $y \in A$ and $\alpha, \beta \in (\Sigma \cup M)^+$. It follows that

$$\sigma x_1 \ldots x_{a(\sigma)}\alpha = y\beta$$

therefore $y \in A \backslash M$. This implies that $y = \tau_A(y_1, \ldots, y_{a(\tau)})$ for some elements $y_1, \ldots, y_{a(\tau)} \in A$. It follows that

$$\sigma x_1 \ldots x_{a(\sigma)}\alpha = \tau y_1 \ldots y_{a(\tau)}\beta$$

therefore $\sigma = \tau$ and $x_1 \ldots x_{a(\sigma)}\alpha = y_1 \ldots y_{a(\tau)}\beta$. Taking into consideration $Q(x_1)$, ..., $Q(x_{a(\sigma)})$ and the fact that $y_1, \ldots, y_{a(\sigma)} \in A$ we obtain $x_i = y_i$ for $i = 1, \ldots, a(\sigma)$ and $\alpha = \beta$. Thus $y = \tau_A(y_1, \ldots, y_{a(\tau)}) = \sigma_A(x_1, \ldots, x_{a(\sigma)})$, therefore $Q(\sigma_A(x_1, \ldots, x_{a(\sigma)}))$ is true. By algebraic induction $Q(x)$ is true for each $x \in A$.

Now we verify the last condition from Definition 2.1.1. Suppose that

$$\sigma_A(x_1, \ldots, x_{a(\sigma)}) = \tau_A(y_1, \ldots, y_{a(\tau)})$$

for some $\sigma, \tau \in \Sigma$ and $x_1, \ldots, x_{a(\sigma)}, y_1, \ldots, y_{a(\tau)} \in A$. It follows that $\sigma x_1 \ldots x_{a(\sigma)}) = \tau y_1 \ldots y_{a(\tau)})$, therefore $\sigma = \tau$ and $x_1 \ldots x_{a(\sigma)} = y_1 \ldots y_{a(\tau)}$. But $Q(x_1)$, ..., $Q(x_{a(\sigma)})$ are true and $y_1, \ldots, y_{a(\tau)} \in A$, therefore $x_1 = y_1$, ..., $x_{a(\sigma)} = y_{a(\sigma)}$. ■

Remark 2.2.1. The proof of the previous proposition relieves the following method by which we obtain a Peano Σ-algebra over the set M:

- Take the Σ-algebra $\mathcal{K} = (K, \{\sigma_K\}_{\sigma \in \Sigma})$, where $K = (\Sigma \cup M)^+$ and for each $\sigma \in \Sigma$:

$$\sigma_K(x_1, \ldots, x_{a(\sigma)}) = \sigma x_1 \ldots x_{a(\sigma)}$$

- Take $A = \overline{M}$, the closure of M in $\mathcal{K}$ and consider the subalgebra $\mathcal{A} = (A, \{\sigma_A\}_{\sigma \in \Sigma})$ of the Σ-algebra $\mathcal{K}$. $\mathcal{A}$ is a Peano Σ-algebra over M.

2.3 The Peano algebras over the same set are isomorphic

In the remaining of this chapter we establish a connection between the Peano Σ-algebra specified in Remark 2.2.1 and other Peano Σ-algebras over the same set.

Proposition 2.3.1. *If $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ and $\mathcal{B} = (B, \{\sigma_B\}_{\sigma\in\Sigma})$ are Σ-algebras free generated by the set M then there is an isomorphism of Σ-algebras $h : A \longrightarrow B$ such that $h(x) = x$ for every $x \in M$.*

Proof. We have $M \subseteq A$ and $M \subseteq B$ because $\mathcal{A}$ and $\mathcal{B}$ are free generated by M.

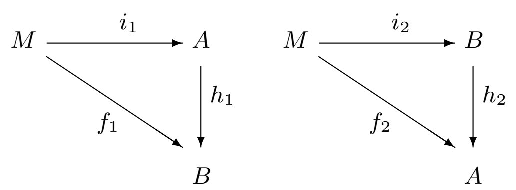


Fig. 2.2. Extensions for f_1 and f_2

We have the situation represented in Figure 2.2 because:

- We take the mapping $f_1 : M \longrightarrow B$ defined by $f_1(x) = x$ for every $x \in M$. But $\mathcal{A}$ is free generated by M and $\mathcal{B}$ is a Σ-algebra therefore the mapping f_1 can be extended to a morphism $h_1 : A \longrightarrow B$.
- Similarly, from the mapping $f_2 : M \longrightarrow A$ defined by $f_2(x) = x$ for every $x \in M$ we obtain a morphism $h_2 : B \longrightarrow A$ such that $f_2 \prec h_2$.

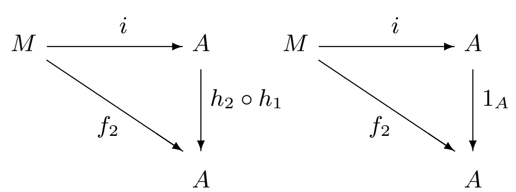


Fig. 2.3. Two extensions of f_2

Applying Proposition 1.4.1 we deduce that $h_2 \circ h_1 : A \longrightarrow A$ is a morphism of Σ-algebras and $h_2 \circ h_1(x) = x$ for every $x \in M$, therefore $f_2 \prec h_2 \circ h_1$.

On the other hand the mapping $1_A : A \longrightarrow A$ is a morphism of Σ-algebras. Thus we have the representation from Figure 2.3. But the mapping f_2 is uniquely extended to a morphism therefore $h_2 \circ h_1 = 1_A$. Similarly, changing the algebra $\mathcal{A}$ with $\mathcal{B}$, we deduce $h_1 \circ h_2 = 1_B$. By Proposition 1.1.1 we deduce that h_1 and h_2 are bijective mappings. Moreover, $f_1 \prec h_1$, therefore $h_1(x) = x$ for every $x \in M$. Thus $h_1 : A \longrightarrow B$ is an isomorphism of Σ-algebras. ∎

Corollary 2.3.1. *Two Peano Σ-algebras $\mathcal{A} = (A, \{\sigma_A\}_{\sigma\in\Sigma})$ and $\mathcal{B} = (B, \{\sigma_B\}_{\sigma\in\Sigma})$ over M are isomorphic algebras. Moreover, there is an isomorphism $h : A \longrightarrow B$ such that $1_M \prec h$.*

Proof. By Proposition 2.1.1 the structures $\mathcal{A}$ and $\mathcal{B}$ are Σ-algebras free generated by M. Now we apply Proposition 2.3.1. ∎

Remark 2.3.1. As a conclusion, if we build the Peano Σ-algebra over M by the method described in Remark 2.2.1 then the structure obtained is isomorphic with every Peano Σ-algebra over the same set M.

Remark 2.3.2. In the theory of labeled stratified graphs and semantic schemas is encountered frequently a particular Peano algebra. This is the case of a singleton $\Sigma = \{\sigma\}$. If M is a finite and nonempty set such that $\sigma \notin M$ then in the first step the following sequence $\{M_n\}_{n\geq 0}$ is obtained:

$$\begin{cases} M_0 = M \\ M_{n+1} = M_n \cup \{\ \sigma uv \mid u \in M_n, v \in M_n\}, \quad n \geq 0 \end{cases}$$

In the second step we take $A = \bigcup_{n\geq 0} M_n$ as support set and the operation $\sigma_A(u,v) = \sigma uv$. We obtain the σ-algebra $\mathcal{A} = (\mathcal{A}, \sigma_{\mathcal{A}})$. This structure is the Peano σ-algebra over M, which is isomorphic with every Peano σ-algebra over the same set M.

3. Lattices and semilattices

The concept of lattice is frequently encountered both in theoretical computer science and in applied computer science but not only. There are two equivalent ways to define this concept: as a poset satisfying additional conditions and by means of the universal algebras. Several computation rules in a lattice are briefly exposed. Finally a concise definition of a very fruitful structure, Boolean algebra, is presented and the basic properties of this structure are described.

3.1 From poset to lattice

Taking the poset as starting point we define in this section the concept of lattice.

Definition 3.1.1. *A poset* $(L, \leq)$ *is a* **lattice** *if for every* $x, y \in L$ *there exists* $sup\{x, y\}$ *and* $inf\{x, y\}$*.*

The above definition allows us to introduce two binary operations on L as we show in the next definition.

Definition 3.1.2. *If* $(L, \leq)$ *is a lattice then we define the following binary operations, named the* **join** (*or* **union** *or* **disjunction**) *and* **meet** (*or* **intersection** *or* **conjunction**) *operations respectively:*

$$\vee : L \times L \longrightarrow L, \; x \vee y = sup\{x, y\} \tag{3.1}$$

$$\wedge : L \times L \longrightarrow L, \; x \wedge y = inf\{x, y\} \tag{3.2}$$

These operations are well defined because the least upper bound and the greatest lower bound are uniquely determined. We observe that they are "total" operations.

Proposition 3.1.1. *The following identities are satisfied in a lattice:*

$$x \vee (y \vee z) = sup\{x, y, z\} \tag{3.3}$$

$$x \wedge (y \wedge z) = inf\{x, y, z\} \tag{3.4}$$

Proof. By the definition of the join operation we have

$$x \vee (y \vee z) = sup\{x, y \vee z\}$$

therefore $x \leq x \vee (y \vee z)$. We observe that we have also

$$y \leq y \vee z \leq x \vee (y \vee z)$$

$$z \leq y \vee z \leq x \vee (y \vee z)$$

therefore $x \vee (y \vee z)$ is an upper bound of $\{x, y, z\}$.
Let us consider another upper bound t_0 of the set $\{x, y, z\}$. But $y \vee z = sup\{y, z\}$ and t_0 is an upper bound of $\{y, z\}$, therefore $y \vee z \leq t_0$ because $y \vee z$ is the least upper bound of $\{y, z\}$. It follows that t_0 is is an upper bound of the set $\{x, y \vee z\}$ and $x \vee (y \vee z) \leq t_0$. In conclusion, $x \vee (y \vee z)$ is the least upper bound of the set $\{x, y, z\}$. In other words, we have (3.3). Similar we prove (3.4). ■

Proposition 3.1.2. *In a lattice* $(L, \leq)$ *the operations* $\vee$ *and* $\wedge$ *satisfy the following identities:*

$$x \vee y = y \vee x, x \wedge y = y \wedge x \ (commutativity) \tag{3.5}$$

$$(x \vee y) \vee z = x \vee (y \vee z), \ (x \wedge y) \wedge z = x \wedge (y \wedge z) \ (associativity) \tag{3.6}$$

$$x \wedge (x \vee y) = x, \ x \vee (x \wedge y) = x \ (absorption) \tag{3.7}$$

Proof. The relations 3.5 are obviously true by (3.1) and (3.2). Based on Proposition 3.1.1 we observe that $x \vee (y \vee z) = sup\{x, y, z\} = sup\{z, x, y\} = z \vee (x \vee y) = (x \vee y) \vee z$. A similar proof is obtained for the associativity of the meet operation, therefore (3.6) is verified.
In order to prove (3.7) we observe that

$$x \leq x \vee (x \wedge y) \tag{3.8}$$

because $x \vee (x \wedge y) = sup\{x, x \wedge y\}$. But $x \wedge y \leq x$ therefore x is an upper bound for the set $\{x, x \wedge y\}$. Thus we can write

$$x \vee (x \wedge y) \leq x \tag{3.9}$$

From (3.8) and (3.9) we obtain $x \wedge (x \vee y) = x$. The other relation from (3.7) is proved in a similar manner. ■

We observe the duality of the relations (3.5), (3.6) and (3.7). According to these relations we remark that the meet and join operations are dual. Moreover, based on (3.6) we can write without any confusion

$$x \vee y \vee z = sup\{x, y, z\} \tag{3.10}$$

and

$$x \wedge y \wedge z = inf\{x, y, z\} \tag{3.11}$$

The next property extends Proposition 3.1.1.

Proposition 3.1.3. *For every finite and nonempty subset M of a lattice $(L, \leq)$ there exists $infM$ and $supM$.*

Proof. Suppose $M = \{x_1, \ldots, x_n\}$ and $n \geq 3$. We prove this property by induction on n. For $n = 2$ the property is obtained directly from Definition 3.1.1, so we supposed $n \geq 3$. By induction on n we verify that

$$x_1 \vee x_2 \vee \ldots \vee x_k = sup\{x_1, x_2, \ldots, x_k\} \tag{3.12}$$

For $k = 3$ the relation (3.12) is true because we have (3.10). Suppose (3.12) is true for $k = n$ and we verify this relation for $k = n+1$. First we prove that $sup\{x_1, \ldots, x_{n+1}\}$ exists and moreover

$$sup\{sup\{x_1, \ldots, x_n\}, x_{n+1}\} = sup\{x_1, \ldots, x_{n+1}\}$$

By the inductive assumption we can denote

$$y = sup\{x_1, \ldots, x_n\} \tag{3.13}$$

The set L is a lattice therefore we have $sup\{y, x_{n+1}\} = z \in L$. Let us verify that $z = sup\{x_1, \ldots, x_{n+1}\}$. From (3.13) we have

$$x_i \leq y \leq z,\ i \in \{1, \ldots, n\} \tag{3.14}$$

and $x_{n+1} \leq z$, therefore z is an upper bound of the set $\{x_1, \ldots, x_{n+1}\}$. Let t be another upper bound of $\{x_1, \ldots, x_{n+1}\}$:

$$x_i \leq t,\ i \in \{1, \ldots, n+1\} \tag{3.15}$$

From (3.15), (3.14) and (3.13) we obtain $y \leq t$. But $x_{n+1} \leq t$, therefore $z = sup\{y, x_{n+1}\} \leq t$.

In conclusion we have

$$x_1 \vee x_2 \vee \ldots \vee x_{n+1} = (x_1 \vee x_2 \vee \ldots \vee x_n) \vee x_{n+1} =$$

$$sup\{sup\{x_1, x_2, \ldots, x_n\}, x_{n+1}\} = sup\{x_1, x_2, \ldots, x_{n+1}\}$$

A similar proof can be given for the relation

$$x_1 \wedge x_2 \wedge \ldots \wedge x_k = inf\{x_1, x_2, \ldots, x_k\}$$

and the proposition is proved. ■

We define now two useful structures. They are used in knowledge representation, especially in knowledge modeling by stratified graphs and semantic schemas.

Definition 3.1.3. *(Rudeanu (1991))*
A **join (meet) semilattice** *is a poset $(L, \leq)$ such that for every $x, y \in L$ there exists $sup\{x, y\}$ $(inf\{x, y\})$.*

a) join semilattice *b*) poset, not a join semilattice

Fig. 3.1. Examples of posets

a) meet semilattice *b*) poset, not a meet semilattice

Fig. 3.2. Examples of posets

Obviously a poset $(L, \leq)$ is a lattice if and only if $(L, \leq)$ is a join and a meet semilattice.

A Hasse diagram can help us to show that a given poset is/not is a semilattice. For example, let us consider the case represented in Figure 3.1. The case a) represents a join semilattice with last element. The case b) gives an example of a poset which is not a semilattice. Obviously this structure can not be a meet semilattice because there isn't any lower bound for $\{a, b\}$. If we try to find $sup\{a, b\}$ then we have to find in the first step the set of the upper bounds of the set $\{a, b\}$. Obviously, this is the set $\{c, d, 1\}$ and the elements c and d are incomparable. As a consequence, it does not exist the least element of this set. In conclusion the element $sup\{a, b\}$ does not exist. A dual case is presented in Figure 3.2.

Obviously we have the property specified in the next proposition.

Proposition 3.1.4. *If in a join (meet) semilattice* $(L, \leq)$ *we define the operation* $\vee : L \times L \longrightarrow L$ $(\wedge : L \times L \longrightarrow L)$ *by* $x \vee y = sup\{x, y\}$ $(x \wedge y = inf\{x, y\})$ *then we obtain an idempotent, commutative and associative operation.*

Remark 3.1.1. (Rudeanu (2001))
The concept of lattice defined in this section is named also *lattice in the sense of Ore*. Shortly we say that this structure is an *Ore lattice*.

Remark 3.1.2. The duality principle holds in lattice theory. Moreover, the concepts of meet semilattice and join semilattice are dual to each other.

3.2 Lattice as universal algebra

In this section we introduce the concept of lattice as universal algebra and we prove that we obtain an equivalent definition.

Definition 3.2.1. *A lattice is an universal algebra* $(L, \vee, \wedge)$ *of signature* $(2, 2)$ *such that the operations satisfy the commutativity (3.5), associativity (3.6) and absorption law (3.7).*

Remark 3.2.1. (Rudeanu (2001)) The concept introduced in Definition 3.2.1 is called also *lattice in the sense of Dedekind*. Shortly we say a *Dedekind lattice*.

Proposition 3.2.1. *In a Dedekind lattice the operations are idempotent:*

$$x \vee x = x, \ \ x \wedge x = x$$

Proof. Applying (3.7) we obtain

$$x \wedge (x \vee x) = x$$

$$x \vee (x \wedge (x \vee x)) = x$$

therefore $x \vee x = x \vee (x \wedge (x \vee x)) = x$. By duality we have the second property $x \wedge x = x$. ■

The following result is a useful property to simplify the introducing of a partial order.

Proposition 3.2.2. *In a Dedekind lattice we have* $x \vee y = y$ *if and only if* $x \wedge y = x$.

Proof. Suppose $x \vee y = y$. We obtain $x \wedge y = x \wedge (x \vee y) = x$ by (3.7). Similar we prove the other implication. ■

We can introduce now a binary relation on a Dedekind lattice.

Proposition 3.2.3. *The following relation defined on a Dedekind lattice is a partial order:*

$$x \leq y \text{ if and only if } x \vee y = y \tag{3.16}$$

Proof. By Proposition 3.2.1 we have $x \vee x = x$, therefore $x \leq x$. Suppose $x \leq y$ and $y \leq x$. Using (3.16) we have $x \vee y = y$ and $y \vee x = x$. By commutativity we obtain $x = y$. To prove the transitivity we suppose $x \leq y$ and $y \leq z$. We have $x \vee y = y$ and $y \vee z = z$, therefore $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$. ■

Proposition 3.2.4. *Every Dedekind lattice* $(L, \vee, \wedge)$ *is an Ore lattice* $(L, \leq)$*, where* $\leq$ *is defined in* (3.16)*.*

Proof. Let us prove that for every $x, y \in L$ there exists $sup\{x, y\}$. Moreover, we prove that $sup\{x, y\} = x \vee y$. We have the following sequence of deductions:

- $x \vee (x \vee y) = (x \vee x) \vee y = x \vee y$ and $y \vee (x \vee y) = (x \vee y) \vee y = x \vee (y \vee y) = x \vee y$, therefore by (3.16) the element $x \vee y$ is an upper bound for the set $\{x, y\}$.
- Let t be an upper bound for $\{x, y\}$. Thus we have $x \leq t$ and $y \leq t$. Using (3.16) we can write $x \vee t = t$ and $y \vee t = t$, therefore $(x \vee y) \vee t = x \vee (y \vee t) = x \vee t = t$. Thus $x \vee y \leq t$.

It follows that $x \vee y$ is the least upper bound of $\{x, y\}$. By duality we prove that $x \wedge y = inf\{x, y\}$. ■

Remark 3.2.2. As we proved in the previous section every Ore lattice is a Dedekind lattice. If in addition we consider Proposition 3.2.4 then we can say that the concepts of Ore lattice and Dedekind lattice are equivalent.

Definition 3.2.2. *Suppose* $\mathcal{L} = (L, \leq)$ *is an Ore lattice. The lattice* $\mathcal{L}_{Ded} = (L, \vee, \wedge)$*, where* $x \vee y = sup_{\leq}\{x, y\}$ *and* $x \wedge y = inf_{\leq}\{x, y\}$ *is called the* **Dedekind lattice associated** *to* $\mathcal{L}$*.*

Definition 3.2.3. *Suppose* $\mathcal{L} = (L, \vee, \wedge)$ *is a Dedekind lattice. The lattice* $\mathcal{L}_{Ore} = (L, \preceq)$*, where* $x \preceq y$ *if and only if* $x \vee y = y$ *is named the* **Ore lattice associated** *to* $\mathcal{L}$*.*

Proposition 3.2.5. *(Grätzer (1971))*

- *If* $\mathcal{L}$ *is an Ore lattice then* $(\mathcal{L}_{Ded})_{Ore} = \mathcal{L}$*.*
- *If* $\mathcal{L}$ *is a Dedekind lattice then* $(\mathcal{L}_{Ore})_{Ded} = \mathcal{L}$*.*

Proof. We prove only the first property. Suppose $\mathcal{L} = (L, \leq)$ is an Ore lattice. We have $\mathcal{L}_{Ded} = (L, \vee, \wedge)$, where

$$x \vee y = sup_{\leq}\{x, y\}, \ x \wedge y = inf_{\leq}\{x, y\}$$

Taking the Ore lattice associated to $\mathcal{L}_{Ded}$ we obtain $(\mathcal{L}_{Ded})_{Ore} = (L, \preceq)$, where

$$x \preceq y \Longleftrightarrow x \vee y = y$$

It follows that:

- If $x \leq y$ then $x \vee y = sup_{\leq}\{x, y\} = y$, therefore $x \preceq y$.
- If $x \preceq y$ then $x \vee y = y$, therefore $sup_{\leq}\{x, y\} = y$. Thus we have $x \leq y$.

In conclusion we have $x \leq$ if and only if $x \preceq y$. ■

In the remainder of this section we treat the concept of semilattice by means of universal algebras.

Definition 3.2.4. (*Grätzer (1971)*)
A **semilattice** *is an algebra* $(L, \circ)$ *of signature* (2) *such that* $\circ$ *is an idempotent, commutative and associative operation.*

Proposition 3.2.6. *Let* $(L, \circ)$ *be a semilattice. We define the following binary relations on* L*:*

$$a \leq b \Longleftrightarrow a \circ b = b \tag{3.17}$$

$$a \sqsubseteq b \Longleftrightarrow a \circ b = a \tag{3.18}$$

Then the following properties are satisfied:

1) $\leq$ *and* $\sqsubseteq$ *are dual relations;*
2) $\leq$ *and* $\sqsubseteq$ *are partial orders;*
3) $(L, \leq)$ *is a join semilattice;*
4) $(L, \sqsubseteq)$ *is a meet semilattice.*

Proof.
1) The dual $\leq_d$ of $\leq$ is defined as follows:

$$x \leq_d \Longleftrightarrow y \leq x$$

Using (3.17) we obtain

$$x \leq_d y \Longleftrightarrow y \circ x = x \tag{3.19}$$

But the operation $\circ$ is commutative therefore from (3.19) we obtain

$$x \leq_d y \Longleftrightarrow x \circ y = x$$

Using (3.18) we obtain

$$x \leq_d y \Longleftrightarrow x \sqsubseteq y$$

and thus the dual of $\leq$ is $\sqsubseteq$.
2) The binary relation $\leq$ is:

• *Reflexive* because the operation $\circ$ is idempotent: $x \circ x = x$, therefore $x \leq x$.

• *Antisymmetric* because from $x \leq y$ and $y \leq x$ we have $x \circ y = y$ and $y \circ x = x$ respectively; but $\circ$ is a commutative operation therefore $x = y$;

• *Transitive* because from $x \leq y$ and $y \leq z$ we obtain $x \circ y = y$ and $y \circ z = z$ respectively; using the associativity of the operation $\circ$ it follows that $x \circ z = x \circ (y \circ z) = (x \circ y) \circ z = y \circ z = z$, therefore $x \leq z$.

The dual of a partial order is a partial order, therefore $\sqsubseteq$ is a partial order.
3) Let us verify that $(L, \leq)$ is a join semilattice. We prove that for every $x, y \in L$ there exists $sup_{\leq}\{x, y\}$. More precisely we show that

$$sup_{\leq}\{x, y\} = x \circ y \tag{3.20}$$

We remark that $x \circ y$ is an upper bound for the set $\{x, y\}$. Really, because $\circ$ is associative and idempotent we have $x \circ (x \circ y) = (x \circ x) \circ y = x \circ y$,

therefore $x \leq x \circ y$. Similar we have $y \leq x \circ y$ because $y \circ (x \circ y) = (y \circ x) \circ y = (x \circ y) \circ y = x \circ (y \circ y) = x \circ y$.
Let t be an arbitrary upper bound for the set $\{x, y\}$. This means that $x \leq t$ and $y \leq t$ therefore $x \circ t = t$ and $y \circ t = t$. We have $(x \circ y) \circ t = x \circ (y \circ t) = x \circ t = t$ therefore $x \circ y \leq t$. Thus $x \circ y$ is the least upper bound of the set $\{x, y\}$ therefore the relation (3.20) is proved.
4) We verify that $L, \sqsubseteq$ is a meet semilattice. To do this we show that for every $x, y \in L$ we have

$$inf_{\sqsubseteq}\{x, y\} = x \circ y$$

First, the element $x \circ y$ is a lower bound for $\{x, y\}$ because:

- $(x \circ y) \circ x = x \circ (y \circ x) = x \circ (x \circ y) = (x \circ x) \circ y = x \circ y$, therefore $x \circ y \sqsubseteq x$;
- $(x \circ y) \circ y = x \circ (y \circ y) = x \circ y$, therefore $x \circ y \sqsubseteq y$.

Let z be a lower bound of the set $\{x, y\}$: $z \sqsubseteq x$ and $z \sqsubseteq y$. From (3.18) we have $z \circ x = z$ and $z \circ y = z$. It follows that $z \circ (x \circ y) = (z \circ x) \circ y = (x \circ z) \circ y = z \circ y = z$, therefore $z \sqsubseteq x \circ y$. ■

Remark 3.2.3. Directly from definition of an Ore lattice we know that if $(L, \leq)$ is such a structure then $(L, \leq)$ is a join semilattice and also $(L, \leq)$ is a meet semilattice. Conversely, if $(L, \leq)$ is both a join semilattice and a meet semilattice then $(L, \leq)$ is an Ore lattice. The last proposition informs us that if $(L, \circ)$ is a semilattice then we can introduce two partial orders $\leq$ and $\sqsubseteq$ such that $(L, \leq)$ is a join semilattice and $(L, \sqsubseteq)$ is a meet semilattice.

3.3 Distributive lattices and complemented lattices

In this section we define two relevant concepts, which allow us to give a concise definition for a fundamental structure of theoretical and applied computer science.

Definition 3.3.1. *A* **distributive lattice** *is a lattice* $(L, \leq)$ *such that the following identities are satisfied:*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{3.21}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{3.22}$$

Proposition 3.3.1. *The identities* (3.21) *and* (3.22) *are equivalent.*

Proof. Suppose (3.21) is true for every $x, y, z \in L$, where L is an arbitrary lattice. Applying (3.21) we obtain:

$$(x \wedge y) \vee (x \wedge z) = [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z] \tag{3.23}$$

therefore by absorption and commutativity we have

$$(x \wedge y) \vee (x \wedge z) = x \wedge [z \vee (x \wedge y)] \tag{3.24}$$

Applying again (3.21), the associativity of the meet operation and the absorption law we obtain

$$x \wedge [z \vee (x \wedge y)] = x \wedge (z \vee x) \wedge (z \vee y) = x \wedge (z \vee y) \tag{3.25}$$

Now from (3.23), (3.24), (3.25) and the commutativity of the join operation we obtain (3.22). The converse implication is obtained by duality. ■

Remark 3.3.1. Applying Proposition 3.3.1 we observe that in Definition 3.3.1 it is enough to suppose that only one of (3.21) or (3.22) is satisfied.

Remark 3.3.2. In the theory of distributive lattices can be used the duality principle because the dual of (3.21) is (3.22) and vice versa.

Definition 3.3.2. *If the lattice* $(L, \leq)$ *has first element and last element then* $(L, \leq)$ *is named* **bounded lattice**. *The first element is denoted by* 0 *and the last element is denoted by* 1. *We suppose* $0 \neq 1$.

Taking into consideration the Dedekind lattice associated to the bounded lattice $(L, \leq)$ we have obviously the following properties:

$0 \leq x \leq 1$ for every $x \in L$;
$x \vee 0 = x$, $x \wedge 0 = 0$ for every $x \in L$;
$x \vee 1 = 1$, $x \wedge 1 = x$ for every $x \in L$;
$x \vee y = 0$ if and only if $x = y = 0$;
$x \wedge y = 1$ if and only if $x = y = 1$.

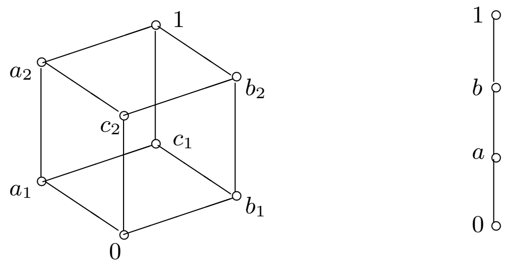
Definition 3.3.3. *Let us consider a bounded lattice* $(L, \leq)$. *An element* $x' \in L$ *is a* **complement** *of the element* $x \in L$ *if the following conditions are satisfied:*

$$\begin{cases} x \wedge x' = 0 \\ x \vee x' = 1 \end{cases} \tag{3.26}$$

A bounded lattice such that every element has at least one complement is a **complemented lattice**.

Remark 3.3.3. In a bounded lattice the following situations can be encountered:

- There is an element which has no complement. This is the case represented in Figure 3.3 case *b*), where the element *a* (as well as *b*) has none complement.
- Some element or every element has just one complement. This is the case presented in Figure 3.3 case *a*).
- One or more elements has/have at least two complements. This case is presented in Figure 1.1, where the element *a* has two complements (the elements *b* and *c*). Moreover, each of the elements *b* and *c* has just one complement, namely *a*.



a) Complemented lattice b) Non-complemented lattice

Fig. 3.3. Hasse diagrams

Proposition 3.3.2. *In a bounded distributive lattice the complement of an element is uniquely determined if exists.*

Proof. Suppose the element x has two complements, x' and x''. We obtain:

$$x' = x' \wedge 1 = x' \wedge (x \vee x'') = (x' \wedge x) \vee (x' \wedge x'') = x' \wedge x''$$

because $x' \wedge x = 0$. Similarly we have

$$x'' = x'' \wedge 1 = x'' \wedge (x \vee x') = (x'' \wedge x) \vee (x'' \wedge x') = x'' \wedge x' = x' \wedge x''$$

therefore $x' = x''$. ■

3.4 Boolean algebras

We can introduce now in a very concise manner a fruitful concept in computer science. This is the subject of the next definition.

Definition 3.4.1. *A distributive and complemented lattice is a* **Boolean algebra***.*

Because in a Boolean algebra each element has a complement and only one, the complement of x is denoted by $\overline{x}$.

Example 3.4.1. The structure represented in Figure 3.3 a) is a Boolean algebra. We consider the set $A = \{a, b, c\}$. If we take the set 2^A as the support set and the set theoretical operations of union and intersection then we obtain a Boolean algebra. The zero element is $\emptyset$ and the element 1 is A. The complement of $X \in 2^A$ is $\overline{X}=A \setminus X$. It is not difficult to observe that this algebra is isomorphic with the algebra represented in Figure 3.3 a).

In what follows we denote a Boolean algebra by the system$(B, \vee, \wedge, ^{-}, 0, 1)$ considered as an algebra of signature $(2,2,1,0,0)$, where $x \vee y = sup\{x, y\}$ and $x \wedge y = inf\{x, y\}$.

We can now relieve some basic properties in a Boolean algebra. For simplicity we shall use the classical notation $x \cdot y = x \wedge y$ and moreover, xy instead of $x \cdot y$.

Proposition 3.4.1. *Let $(B, \vee, \wedge, ^{-}, 0, 1)$ be a Boolean algebra. The following properties are satisfied and can be viewed as computation rules on a Boolean algebra:*

1. $x = y \Longleftrightarrow x \vee t = y \vee t$ *for every* $t \in B$;
2. $x = y \Longleftrightarrow x \cdot t = y \cdot t$ *for every* $t \in B$;
3. $x \vee y = 0 \Longleftrightarrow x = y = 0$;
4. $x \cdot y = 1 \Longleftrightarrow x = y = 1$;
5. $\overline{x \vee y} = \overline{x} \cdot \overline{y}$,
   $\overline{x \cdot y} = \overline{x} \vee \overline{y}$ *(De Morgan laws);*
6. $\overline{\overline{x}}=x$ *(double negation law);*
7. $x \leq y \Longrightarrow \overline{y} \leq \overline{x}$;
8. $x \leq y \Longleftrightarrow x \vee t \leq y \vee t$ *for every* $t \in B$;
9. $x \leq y \Longleftrightarrow x \cdot t \leq y \cdot t$ *for every* $t \in B$;
10. $x \cdot (\overline{x} \vee y) = x \cdot y$;
    $x \vee (\overline{x} \cdot y) = x \vee y$; *(Boolean absorption)*
11. $x \leq y \Longleftrightarrow x \cdot \overline{y} = 0 \Longleftrightarrow \overline{x} \vee y = 1$.
12. $x = y \Longleftrightarrow x \cdot \overline{y} \vee \overline{x} \cdot y = 0$

Proof.

1. If $x \vee t = y \vee t$ for every $t \in B$ then particularly for $t = 0$ we obtain $x = y$.
2. Similarly, take $t = 1$.
3. If $x \vee y = 0$ then $x = y = 0$ because $x \leq x \vee y$, $y \leq x \vee y$ and 0 is the first element.
4. Immediate because $x \geq x \cdot y = 1$ and 1 is the last element.
5. $(x \vee y) \vee (\overline{x} \cdot \overline{y}) = (x \vee y \vee \overline{x}) \cdot (x \vee y \vee \overline{y}) = (y \vee 1) \cdot (x \vee 1) = 1$; $(x \vee y) \cdot (\overline{x} \cdot \overline{y}) = (\overline{x} \cdot \overline{y}) \cdot (x \vee y) = (\overline{x} \cdot \overline{y} \cdot x) \vee (\overline{x} \cdot \overline{y} \cdot y) = 0 \vee 0 = 0$;
6. We have $x \vee \overline{x} = 1$, $x \cdot \overline{x} = 0$ because $\overline{x}$ is the complement of x. But $\overline{\overline{x}}$ is the complement of $\overline{x}$ therefore $\overline{x} \vee \overline{\overline{x}} = 1$ and $\overline{\overline{x}} \cdot \overline{x} = 0$. These relations are interpreted as follows: x and $\overline{\overline{x}}$ are the complements of $\overline{x}$. The complement is uniquely determined therefore $x = \overline{\overline{x}}$.
7. $\overline{y} \leq \overline{x} \Longleftrightarrow \overline{y} \cdot \overline{x} = \overline{y} \Longleftrightarrow \overline{\overline{y} \cdot \overline{x}} = \overline{\overline{y}} \Longleftrightarrow y \vee x = y \Longleftrightarrow x \leq y$;
8. $x \leq y \Longrightarrow x \vee y = y \Longrightarrow x \vee y \vee t = y \vee t$; for the converse implication we choose $t = 0$.
9. Similar to 8.
10. By distributivity we have $x \cdot (\overline{x} \vee y) = (x \cdot \overline{x}) \vee (x \cdot y) = 0 \vee x \cdot y = x \cdot y$. The name of this law comes from the fact $\overline{x}$ is absorbed.

11. If $x \leq y$ then $x \cdot \overline{y} \leq y \cdot \overline{y} = 0$. Conversely, if $x \cdot \overline{y} = 0$ then $(x \cdot \overline{y}) \vee y = y$, therefore $(x \vee y) \cdot (\overline{y} \vee y) = y$. Thus $x \vee y = y$, therefore $x \leq y$. Further we obtain by De Morgan law and double negation $x \cdot \overline{y} = 0 \Longleftrightarrow \overline{x} \vee y = 1$.
12. The direct implication is immediate. If $x \cdot \overline{y} \vee \overline{x} \cdot y = 0$ then $x \cdot \overline{y} = \overline{x} \cdot y = 0$, therefore $x \leq y$ and $y \leq x$.

Proposition 3.4.2. *Every Boolean algebra* $(B, \vee, \wedge, ^-, 0, 1)$ *is a ring with unit* $(B, +, \cdot, -, 0, 1)$*, where* $x + y = (x \wedge \overline{y}) \vee (\overline{x} \vee y)$ *and* $x \cdot y = x \wedge y$.

Proof. A routine computation shows that all axioms of a ring are satisfied and $x + x = 0$, therefore $-x = x$. ∎

Remark 3.4.1. The problem to find the simplest example of a given algebraic structure is frequently encountered. Particularly this problem can appear also in the case of lattices or Boolean algebras. Developing this idea, we can relieve the following aspects:

1. The simplest lattice is the structure $(\{a\}, \leq)$ of one element, with the partial order given by $a \leq a$. This structure is called the *degenerate lattice*. Obviously all degenerate lattices are isomorphic. A lattice containing at least two elements is called a *non-degenerate lattice*. The simplest non-degenerate lattice is $\mathcal{L} = (\{a, b\}, \leq)$, where $infL = a$ and $supL = b$. Thus the simplest non-degenerate lattice is a bounded lattice.
2. The simplest Boolean algebra is the structure $(\{a\}, \vee, \wedge, ^-, a, a)$, where $a \vee a = a \wedge a = a$ and $a = \overline{a}$. This structure is named also the *degenerate Boolean algebra*. A Boolean algebra is a *non-degenerate Boolean algebra* if it contains at least two elements. The simplest non-degenerate Boolean algebra is also named the *binary Boolean algebra* and this structure is usually denoted by $\mathcal{B}_2 = (\{0, 1\}, \vee, \wedge, ^-, 0, 1)$ and its operations are defined in Table 3.1.

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

Table 3.1. The operations of $\mathcal{B}_2$

Although from the point of view of the general theory of Boolean algebra the structure given by $\mathcal{B}_2$ is very simple, from the point of view of the applications in binary logic and combinational circuits this structure establishes an essential support.

3.5 Boolean rings

In the previous section we shown that every Boolean algebra can be organized as a ring with unit. This structure enjoys of some specific properties, which are presented in this section.

Definition 3.5.1. *A ring with unit* $\mathcal{R} = (R, +, \cdot, -, 0, 1)$ *is a* **Boolean ring** *if the following condition is satisfied*

$$x \cdot x = x \tag{3.27}$$

for every $x \in R$.

Proposition 3.5.1. *If* $\mathcal{R} = (R, +, \cdot, -, 0, 1)$ *is a Boolean ring then the following identities are satisfied:*

$$x + x = 0 \tag{3.28}$$

$$x \cdot y = y \cdot x \tag{3.29}$$

Proof. Consider an arbitrary element $x \in R$. Based on (3.27) we have

$$(x + x)(x + x) = x + x$$

and by distributivity we obtain

$$x \cdot x + x \cdot x + x \cdot x + x \cdot x = x + x \tag{3.30}$$

Taking into account the property (3.27) in (3.30) we obtain

$$x + x + x + x = x + x$$

therefore (3.28) is proved.
Take another element $y \in R$. Based on (3.27) we have

$$(x + y) \cdot (x + y) = x + y$$

therefore

$$x \cdot x + x \cdot y + y \cdot x + y \cdot y = x + y$$

It follows that

$$x \cdot y + y \cdot x = 0 \tag{3.31}$$

But from (3.28) we have

$$x \cdot y + x \cdot y = 0 \tag{3.32}$$

From (3.31) and (3.32) we obtain

$$x \cdot y + y \cdot x = x \cdot y + x \cdot y$$

therefore (3.29) is proved. ∎

Corollary 3.5.1. *In a Boolean ring* $-x = x$.

Proof. Really, from (3.28) we obtain $-x = x$.

Proposition 3.5.2. *Every Boolean algebra* $\mathcal{B} = (B, \vee, \wedge, ^-, 0, 1)$ *is a Boolean ring* $\mathcal{B}_\otimes = (B, +, \cdot, -, 0, 1)$ *where*

$$x + y = (x \wedge \overline{y}) \vee (\overline{x} \wedge y)$$

$$x \cdot y = x \wedge y$$

$$-x = x$$

Proof. Immediate by Proposition 3.4.2 and Definition 3.5.1. ∎

Proposition 3.5.3. *Every Boolean ring with unit* $\mathcal{B} = (B, +, \cdot, -, 0, 1)$ *is a Boolean algebra* $\mathcal{B}_\triangle = (B, \vee, \wedge, ^-, 0, 1)$ *where*

$$x \vee y = x + y + x \cdot y$$

$$x \wedge y = x \cdot y$$

$$\overline{x} = x + 1$$

Proof. Immediate by a routine computation. For example, $x \vee \overline{x} = x + (x + 1) + x \cdot (x + 1) = 1 + x \cdot x + x = 1 + x + x = 1$ and $x \wedge \overline{x} = x \cdot (x + 1) = x \cdot x + x = x + x = 0$. ∎

Proposition 3.5.4. *If* $\mathcal{B}$ *is a Boolean algebra then* $(\mathcal{B}_\otimes)_\triangle = \mathcal{B}$.

Proof. If $\mathcal{B} = (B, \vee, \wedge, ^-, 0, 1)$ then

$$\mathcal{B}_\otimes = (B, +, \cdot, -, 0, 1)$$

where

$$x + y = (x \wedge \overline{y}) \vee (\overline{x} \wedge y)$$

$$x \cdot y = x \wedge y$$

$$-x = x$$

Now if we take

$$(\mathcal{B}_\otimes)_\triangle = (B, \sqcup, \sqcap, ^\star, 0, 1)$$

then

$$x \sqcup y = x + y + x \cdot y$$

$$x \sqcap y = x \cdot y$$

$$x^\star = x + 1$$

We observe that

$$x^\star = 1 + x = (1 \wedge \overline{x}) \vee (\overline{1} \wedge x) = \overline{x} \vee 0 = \overline{x}$$

and

$$x \sqcap y = x \cdot y = x \wedge y$$

Moreover,

$$x \sqcup y = x + y + x \cdot y = x + y \cdot (1 + x) = x + y \cdot x^\star =$$
$$x + y \cdot \overline{x} = [x \wedge \overline{y \wedge \overline{x}}] \vee (\overline{x} \wedge y \wedge \overline{x}) =$$
$$[(x \wedge (\overline{y} \vee x)] \vee (\overline{x} \wedge y) = (x \wedge \overline{y}) \vee (x \wedge x) \vee (\overline{x} \wedge y) =$$
$$x \vee (\overline{x} \wedge y) = x \vee y$$

■

Proposition 3.5.5. *If $\mathcal{B}$ is a Boolean ring with unit then $(\mathcal{B}_\triangle)_\otimes = \mathcal{B}$.*

Proof. Consider a Boolean ring with unit $\mathcal{B} = (B, +, \cdot, -, 0, 1)$. The structure

$$\mathcal{B}_\triangle = (B, \vee, \wedge, ^-, 0, 1)$$

has the following operations:

$$x \vee y = x + y + x \cdot y$$
$$x \wedge y = x \cdot y$$
$$\overline{x} = x + 1$$

If further we take the structure

$$(\mathcal{B}_\triangle)_\otimes = (B, \oplus, \odot, \ominus, 0, 1)$$

then

$$x \oplus y = (x \wedge \overline{y}) \vee (\overline{x} \wedge y)$$
$$x \odot y = x \wedge y$$
$$\ominus x = x$$

We obtain

$$x \oplus y = (x \wedge \overline{y}) \vee (\overline{x} \wedge y) = [x \wedge (y + 1)] \vee [(x + 1) \wedge y] =$$
$$[x \cdot (y + 1)] \vee [(x + 1) \cdot y] = [(x \cdot y) + x] \vee [(x \cdot y) + y] =$$
$$[(x \cdot y) + x] + [(x \cdot y) + y] + [(x \cdot y) + x] \cdot [(x \cdot y) + y] =$$
$$(x \cdot y) + x + (x \cdot y) + y + (x \cdot y) \cdot (x \cdot y) + (x \cdot y) \cdot y + x \cdot (x \cdot y) + x \cdot y =$$
$$[(x \cdot y) + (x \cdot y)] + (x + y) + (x \cdot y) + x \cdot (y \cdot y) + (x \cdot x) \cdot y + x \cdot y =$$
$$0 + (x + y) + (x \cdot y) + (x \cdot y) + (x \cdot y) + (x \cdot y) = 0 + (x + y) + 0 + 0 = x + y$$

For the second operation we obtain

$$x \odot y = x \wedge y = x \cdot y$$

and for the third operation we have

$$\ominus x = x = -x$$

therefore the proposition is proved. ■

Proposition 3.5.6. *Consider a Boolean algebra* $\mathcal{B} = (B, \vee, \wedge, ^-, 0, 1)$ *and the Boolean ring with unit* $\mathcal{B}_{\otimes} = (B, +, \cdot, -, 0, 1)$ *associated to* $\mathcal{B}$. *The following identities are satisfied:*

i) $x + y = x \vee y \Longleftrightarrow x \wedge y = 0$;
ii) $x = y \Longleftrightarrow x + y = 0$;
iii) $x + \overline{x} \cdot y = x \vee y$.

Proof.
i) We have $x + y = x \vee y$ if and only if

$$x \cdot \overline{y} \vee \overline{x} \cdot y = x \vee y \tag{3.33}$$

If each member of (3.33) is multiplied by x we obtain $x \cdot \overline{y} = x \vee x \cdot y$. But $x \vee x \cdot y = x$ therefore $x \cdot \overline{y} = x$. This implies $x \leq \overline{y}$ and thus we have $x \wedge y = 0$.
Conversely, suppose $x \wedge y = 0$. It follows that $x \leq \overline{y}$ and $y \leq \overline{x}$. Equivalently we can write $x \cdot \overline{y} = x$ and $\overline{x} \cdot y = y$. But $x + y = x \cdot \overline{y} \vee \overline{x} \cdot y$. Thus we have $x + y = x \vee y$.
ii) By Proposition 3.4.1 we have $x = y$ if and only if $x \cdot \overline{y} \vee \overline{x} \cdot y = 0$. Thus $x = y$ if and only if $x + y = 0$.
iii) We have $x + \overline{x} \cdot y = x \cdot \overline{\overline{x} \cdot y} \vee \overline{x} \cdot \overline{x} \cdot y = x \cdot (x \vee \overline{y}) \vee \overline{x} \cdot y = x \vee \overline{x} \cdot y = x \vee y$.
■

Remark 3.5.1. The previous results establish a bijection between the class of Boolean algebras and the class of Boolean rings with unit.

Example 3.5.1. Consider the particular case of the Boolean algebra $\mathcal{M} = (2^M, \cup, \cap, C, \emptyset, M)$ endowed with the set theoretical operations

$$A \cup B = \{x \in M \mid x \in A \textit{ or } x \in B\}$$

$$A \cap B = \{x \in M \mid x \in A \textit{ and } x \in B\}$$

$$C(A) = M \setminus A$$

Then $\mathcal{M}_{\otimes} = (2^M, +, \cdot, -, 0, 1)$ where

$$A + B = (A \cap C(B)) \cup (B \cap C(A)) = (A \setminus B) \cup (B \setminus A)$$

is the operation known as the *symmetric difference* of two sets. In addition we have $-A = A$ for every subset A of M.

Remark 3.5.2. The concepts of Boolean algebras and Boolean rings are strongly connected. This can be observed by the fact that various results from Boolean algebra are preserved in Boolean rings and vice versa. An example of such a property is the following: if $\mathcal{B}_1$ and $\mathcal{B}_2$ are Boolean algebras and $h : \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a morphism of Boolean algebras then $h : (\mathcal{B}_1)_{\otimes} \longrightarrow (\mathcal{B}_2)_{\otimes}$ is a morphism of Boolean rings.

4. Case studies and related problems

This chapter can be considered as a complement of the previous chapters, an illustration of the computation rules and the basic concepts for various algebraic structures taken as universal algebras.
There are two objectives of this chapter:

- to emphasize a possible benefit obtained by the use of universal algebras theory.
- to present several results that can hardly be introduced within the previous chapters of this volume but these results help us to obtain short solutions for various problems.

The reader can easy observe that some concepts as sublattice, subsemilattice, Boolean subalgebra and other substructures are not developed in the previous chapters. In this chapter we show that these substructures can be taken from the general theory of universal algebras. Several properties for these particular algebraic structures are relieved.

4.1 Lattices

In this section we apply some general concepts of universal algebras such as free algebras and subalgebras to lattice theory. In addition, because the concept of distributive lattice is a main concept, we present several useful properties of this structure. Among these properties we distinguish a visual condition by which we can decide on a Hasse diagram of a lattice whether or not the corresponding lattice is a distributive one.

4.1.1 Free lattices

The concept of Σ-algebra free generated by a set, introduced in Chapter 1, was used in the study of Peano algebras as we shown in Chapter 2. In this section we relieve another aspect concerning the applications of this concept. We consider the following general problem: *find the most general lattice that can be formed satisfying some conditions.*
In order to illustrate the treatment of such a problem we develop an interesting idea taken from Grätzer (1971): *find the most general lattice generated by the set* $\{a, b, c\}$ *such that* $b < a$.

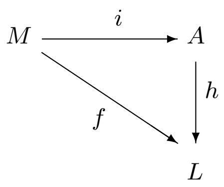


Fig. 4.1. The extension of f

We denote by $\mathcal{A} = (A, \vee, \wedge)$ the lattice satisfying the conditions imposed above and consider $M = \{a, b, c\}$. We take

$$M_1 = M_0 \cup \{a \vee c, b \vee c, a \wedge c, b \wedge c\}$$

as in (1.3), where $M_0 = M$.
We discern the following steps:
1) The elements of M_1 are pairwise distinct.
In order to prove this assertion we use Definition 1.4.3. For every lattice $\mathcal{L} = (L, \sqcup, \sqcap)$, every mapping $f : M \longrightarrow L$ can be extended to a morphism $h : A \longrightarrow L$ as in Figure 4.1. Particularly we can apply this property for the lattice $\mathcal{L}$=$(\{0, 1, 2\}, \leq)$, where $x \sqcup y = max\{x, y\}$ and $x \sqcap y = min\{x, y\}$. We choose the mapping $f : M \longrightarrow B$ defined by $f(b) = 0,\ f(a) = 1,\ f(c) = 2$.
Based on this choice we can verify that

$$a \neq a \vee c,\ \ b \neq a \vee c$$

$$b \neq a \wedge c,\ \ a \wedge c \neq b \wedge c$$

$$a \vee c \neq b \wedge c,\ \ a \vee c \neq a \wedge c$$

$$b \vee c \neq a \wedge c,\ \ a \vee c \neq b \wedge c$$

For instance, if by contrary we suppose $a = a \vee c$ then $h(a) = h(a) \sqcup h(c) = f(b) \sqcup f(c) = 0 \sqcup 2 = 2$, which is not possible because $h(a) = f(a) = 1$.
If we choose the mapping $f(b) = 1,\ f(a) = 2$ and $f(c) = 0$ then we can verify that $a \neq a \wedge c$, $c \neq a \vee c$ and $a \vee c \neq b \wedge c$.
2) If we continue to compute M_2 as in (1.3) then we have to consider the elements $a \wedge (b \vee c)$ and $b \vee (a \wedge c)$. Until now we obtained the situation presented in Figure 4.2.
Let us prove that

$$a \wedge (b \vee c) \geq b \vee (a \wedge c) \tag{4.1}$$

From $a \geq b$ and $b \vee c \geq b$ we deduce that

$$a \wedge (b \vee c) \geq b \tag{4.2}$$

We have also $a \geq a$ and $b \vee c \geq c$, therefore

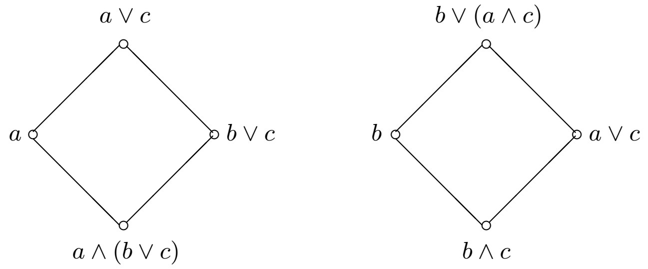


Fig. 4.2. Hasse diagrams

$$a \wedge (b \vee c) \geq a \wedge c \tag{4.3}$$

Now, from (4.2) and (4.3) we obtain (4.1).
We verify now that

$$(b \vee (a \wedge c)) \wedge c = a \wedge c \tag{4.4}$$

From $b \vee (a \wedge c) \geq a \wedge c$ and $c \geq a \wedge c$ we deduce $b \vee (a \wedge c) \geq a \wedge c$, therefore

$$(b \vee (a \wedge c)) \wedge c \geq a \wedge c \tag{4.5}$$

and from $b \leq a$ we obtain $b \vee (a \wedge c) \leq a \vee (a \wedge c) = a$, therefore

$$(b \vee (a \wedge c)) \wedge c \leq a \wedge c \tag{4.6}$$

Now (4.4) is obtained from (4.5) and (4.6). As a conclusion we obtained the situation presented in Figure 4.3.
If we try to find other elements by joins and meets we observe that each of these elements is equal to a given one. The reader can easy verify that the Hasse diagram from Figure 4.3 gives a lattice of nine elements satisfying the imposed conditions.

4.1.2 Sublattices

The reader can observe that in Chapter 3 dedicated to lattices we didn't define the concept of sublattice of a lattice. This is explained by the fact that a lattice is an universal algebra and by Definition 1.3.5 we introduced the concept of subalgebra of a algebra as well as the subalgebra generated by a subset of an algebra.

Example 4.1.1. We consider the set $N = \{0, 1, \ldots\}$ of all natural numbers and the relation $div \subseteq N \times N$ defined as follows: $x \; div \; y$ if an only if x is a divisor of y. In other words, $x \; divy$ if there exists a natural number k such that $y = x \cdot k$. The relation div is reflexive, antisymmetric and transitive, therefore it is a partial order. We obtain the poset $\mathcal{N} = (N, div)$. Moreover, $\mathcal{N}$ is a lattice because:

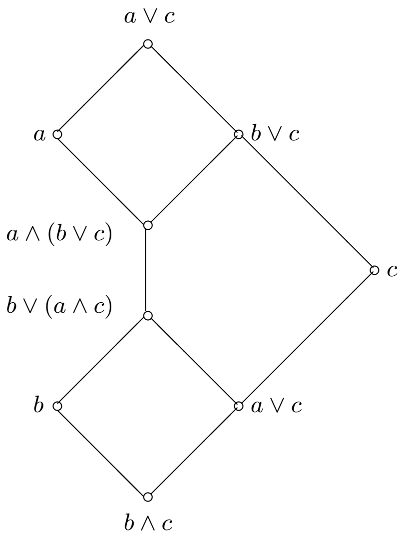


Fig. 4.3. The lattice A

$$inf\{x, y\} = g.c.d(x, y)$$

$$sup\{x, y\} = l.c.m(x, y)$$

where $g.c.d(x, y)$ means "the greatest common divisor" and $l.c.m(x, y)$ means the "least common multiple" of the corresponding numbers. Thus $\mathcal{N}$ becomes an Ore lattice.

We consider the Dedekind lattice $\mathcal{N}_{Ded} = (N, \vee, \wedge)$ and the problem is *to find the least lattice of natural numbers with respect to div, which contains the numbers* 2, 3 *and* 5. Equivalently this problem can be restated in the language of universal algebras as follows:

Consider the set $M = \{2, 3, 5\}$ and the algebra $\mathcal{N}_{Ded} = (N, \vee, \wedge)$. Find the subalgebra (equivalently, the sublattice) generated by M in $\mathcal{N}$.

In order to solve this problem we apply Proposition 1.3.4 to obtain the closure of M in $\mathcal{N}_{Ded}$:

$M_0 = \{2, 3, 5\}$;
$M_1 = M_0 \cup \{1, 6, 10, 15\}$;
$M_2 = M_1 \cup \{30\}$;
$M_3 = M_2$

It follows that $\overline{M} = \{1, 2, 3, 5, 6, 10, 15, 30\}$. Because $\mathcal{M}$ is a finite set, the operations of the sublattice $\mathcal{M} = (\overline{M}, \vee, \wedge)$ can be represented as in Table 4.1 and Table 4.2.

These two tables give a complete description of the operations from $\mathcal{M}$. But we have also a graphical method to represent a poset, particularly

sup	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	2	2	6	10	6	10	30	30
3	3	6	3	15	6	30	15	30
5	5	10	15	5	30	30	15	30
6	6	6	6	30	6	30	30	30
10	10	10	30	30	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

Table 4.1. The operation sup

inf	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	6	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

Table 4.2. The operation inf

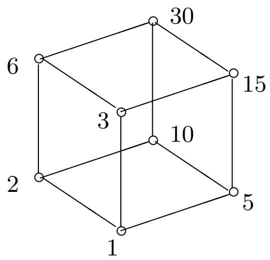


Fig. 4.4. Hasse diagram for $\mathcal{M}$

a lattice. This is the Hasse diagram and for $\mathcal{M}$ this representation is given in Figure 4.4. The reader can compare the representation method by tables and by Hasse diagrams.

A slight version of the previous problem is presented in the next example.

Example 4.1.2. We consider the set $N = \{0, 1, \ldots\}$ of all natural numbers and the relation $div \subseteq N \times N$. As we observed in the previous example, the structure $\mathcal{N} = (N, div)$ is a poset. Moreover, $\mathcal{N}$ is an Ore lattice and $inf\{x, y\} = g.c.d(x, y)$, $sup\{x, y\} = l.c.m(x, y)$. Thus $\mathcal{N}$ is both a join semilattice and a meet semilattice with respect to div. We consider the following problem, which is immediately solved:
Find the join subsemillatice $\mathcal{N}_j$ and the meet subsemilattice $\mathcal{N}_m$ generated by the set $M = \{2, 3, 5\}$.

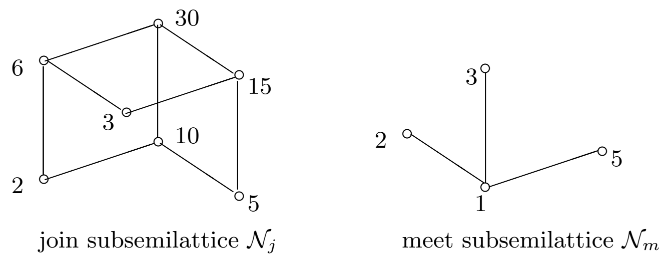


Fig. 4.5. Subsemilattices generated by M

Applying Proposition 1.3.4 we obtain the closure of M with respect to the operation $\vee$:

$M_0 = \{2, 3, 5\}$;
$M_1 = M_0 \cup \{6, 10, 15\}$;
$M_2 = M_1 \cup \{30\}$;
$M_3 = M_2$

It follows that $\overline{M} = \{2, 3, 5, 6, 10, 15, 30\}$ and the join subsemilattice $\mathcal{N}_j$ is represented in Figure 4.5.
For the closure of M with respect to $\wedge$ we obtain

$M_0 = \{2, 3, 5\}$;
$M_1 = M_0 \cup \{1\}$;
$M_2 = M_1$

and the corresponding Hasse diagram for $\mathcal{N}_m$ is drawn also in Figure 4.5.

4.1.3 Distributive lattices

The next two results help us to obtain a useful characterization for the distributive lattices.

Proposition 4.1.1. *The following inequality holds in any lattice:*

$$(x \vee y) \wedge (x \vee z) \geq x \vee (y \wedge (x \vee z)) \tag{4.7}$$

Proof. Obviously we have $x \vee y \geq x$ and $x \vee z \geq x$ therefore

$$(x \vee y) \wedge (x \vee z) \geq x \tag{4.8}$$

We have also

$$x \vee y \geq y \wedge (x \vee z) \tag{4.9}$$

because $x \vee y \geq y \geq y \wedge (x \vee z)$ and

$$x \vee z \geq y \wedge (x \vee z) \tag{4.10}$$

because $(x \vee z) \wedge (y \wedge (x \vee z)) = y \wedge (x \vee z)$.
From (4.9) and (4.10) we obtain

$$(x \vee y) \wedge (x \vee z) \geq y \wedge (x \vee z) \tag{4.11}$$

and from (4.8) and (4.11) we obtain (4.7). ■

Corollary 4.1.1. *In any lattice we have*

$$(x \vee y) \wedge (x \vee z) \geq x \vee (y \wedge z) \tag{4.12}$$

Proof. From (4.7) we have

$$(x \vee y) \wedge (x \vee z) \geq x \vee (y \wedge (x \vee z)) \geq x \vee (y \wedge z)$$

because $x \vee z \geq z$ and $y \wedge (x \vee z) \geq y \wedge z$. ■

Proposition 4.1.2. *(Grätzer (1971))*
A lattice $\mathcal{L} = (L, \vee, \wedge)$ is distributive if and only if the following inequality is satisfied for every $x, y, z \in L$:

$$(x \vee y) \wedge z \leq x \vee (y \wedge z) \tag{4.13}$$

Proof. If $\mathcal{L}$ is a distributive lattice then

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

therefore (4.13) is true since $x \vee z \geq z$.
Conversely, suppose (4.13) is verified. Take the arbitrary elements $a, b, c \in L$. If we apply (4.13) for $x = a$, $y = b$ and $z = a \vee c$ then we obtain

$$(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge (a \vee c)) = a \vee ((a \vee c) \wedge b) \tag{4.14}$$

Taking $x = a$, $y = c$, $z = b$ in (4.13) we obtain

$$(a \vee c) \wedge b \leq a \vee (c \wedge b) = a \vee (b \wedge c) \tag{4.15}$$

From (4.14) and (4.15) we obtain

$$(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge (a \vee c)) \leq a \vee (a \vee (b \wedge c)) \tag{4.16}$$

But

$$a \vee (a \vee (b \wedge c)) = (a \vee a) \vee (b \wedge c) = a \vee (b \wedge c)$$

therefore from (4.16) we obtain

$$(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c) \tag{4.17}$$

Applying (4.12) we obtain

$$(a \vee b) \wedge (a \vee c) \geq a \vee (b \wedge c) \tag{4.18}$$

therefore from (4.17) and (4.18) we have

$$(a \vee b) \wedge (a \vee c) = a \vee (b \wedge c)$$

In other words, the lattice L is distributive. ∎

Proposition 4.1.3. *(Grätzer (1971))*
In any lattice the following properties are equivalent:

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee (x \wedge z)) \tag{4.19}$$

$$x \geq z \Longrightarrow (x \wedge y) \vee z = x \wedge (y \vee z) \tag{4.20}$$

Proof. Suppose (4.19) is satisfied. If $x \geq z$ then $x \wedge z = z$ therefore (4.19) becomes $(x \wedge y) \vee z = x \wedge (y \vee z)$. Conversely, suppose that (4.20) is satisfied. Consider the arbitrary elements a, b and c and take $x = a$, $y = b$ and $z = a \wedge c$ in (4.20). We obtain

$$(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee (a \wedge c))$$

and thus (4.19) is verified. ∎

Definition 4.1.1. *A lattice satisfying (4.19) or (4.20) is named* **modular lattice**.

Remark 4.1.1. The property (4.19) or (4.20) is named *modular law.*

An useful property is presented in the next proposition.

Proposition 4.1.4. *A lattice $\mathcal{L} = (L, \vee, \wedge)$ is modular if and only if*

$$x \leq z \Longrightarrow x \vee (y \wedge z) \geq (x \vee y) \wedge z \tag{4.21}$$

for every $x, y, z \in L$.

Proof. The direct implication is obtained immediately. If the lattice is modular then by (4.20) we have

$$z \geq x \Longrightarrow (z \wedge y) \vee x = z \wedge (y \vee x)$$

therefore (4.21) is verified.
Conversely, if $x \leq z$ then from (4.12) and (4.21) we deduce

$$(x \vee y) \wedge z \geq x \vee (y \wedge z) \geq (x \vee y) \wedge z$$

therefore $(x \vee y) \wedge z = x \vee (y \wedge z)$. In conclusion, if (4.21) is satisfied then

$$z \geq x \Longrightarrow (x \vee y) \wedge z = x \vee (y \wedge z)$$

and by (4.20) and Definition 4.1.1 the lattice is modular. ∎

Proposition 4.1.5. *Every distributive lattice is modular.*

Proof. Suppose $x \geq z$. Because the lattice is distributive we have

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) = x \wedge (y \vee z)$$

therefore the lattice is modular. ∎

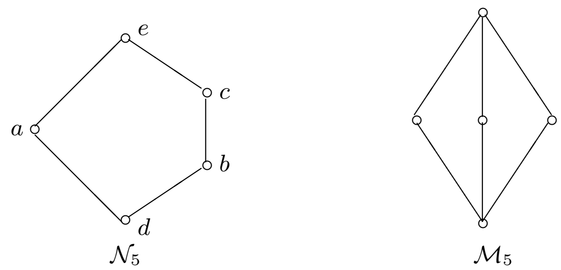


Fig. 4.6. Hasse diagrams for $\mathcal{N}_5$ and $\mathcal{M}_5$

Proposition 4.1.6. *(Burris (1981), Grätzer (1971)) Consider the lattice $\mathcal{N}_5$ depicted in Figure 4.6. The lattice $\mathcal{L}$=$(L, \vee, \wedge)$ is modular if and only if L has no sublattice isomorphic to $\mathcal{N}_5$.*

Proof. Obviously $\mathcal{N}_5$ is not a modular lattice. For instance, in Figure 4.6 we have $b \vee (a \wedge c) = b \vee d = b$ and $(b \vee a) \wedge (b \vee c) = e \wedge c = c$, therefore this lattice is a non distributive one. Thus $\mathcal{L}$ is not a modular lattice.
Conversely, suppose $\mathcal{L}$ is a non modular lattice. By Proposition 4.21 there are $x, y, z \in L$ such that $x \leq z$ and

$$x \vee (y \wedge z) < (x \vee y) \wedge z \tag{4.22}$$

We consider the elements $a = x \vee (y \wedge z)$ and $b = (x \vee y) \wedge z$. From (4.22) we have $a < b$.
Using the elements a and b we shall obtain a sublattice of $\mathcal{L}$ isomorphic to $\mathcal{N}_5$. In order to obtain this sublattice we observe that:

- $y \wedge b = y \wedge [(x \vee y) \wedge z] = [y \wedge (x \vee y)] \wedge z = y \wedge z$;
- $y \vee a = y \vee [x \vee (y \wedge z)] = x \vee y \vee (y \wedge z) = y \vee x$;
- $y \wedge z \leq a$ because $a = x \vee (y \wedge z)$;
- $b \leq y \vee x$ because $b = (x \vee y) \wedge z$;
- $y \wedge z \leq a < b$ therefore $y \wedge z = y \wedge (y \wedge z) \leq y \wedge a \leq y \wedge b = y \wedge z$ and thus $y \wedge z = y \wedge b = y \wedge a$;
- $y \vee x \geq b$ therefore $y \vee x = y \vee (y \vee x) \geq y \vee b \geq y \vee a = y \vee x$

and thus $y \vee x = y \vee a = y \vee b$.

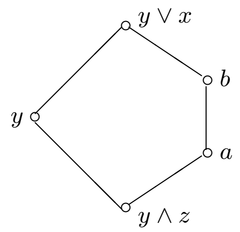


Fig. 4.7. Hasse diagram

The Hasse diagram for the set $\{a, b, y, y \wedge z, y \vee x\}$ is depicted in Figure 4.7. We obtained a sublattice of $\mathcal{L}$ that is isomorphic to $\mathcal{N}_5$ and the proposition is proved. ■

Proposition 4.1.7. *(Grätzer (1971), Burris (1981)) Consider the lattice $\mathcal{M}_5$ depicted in Figure 4.6. A modular lattice is distributive if and only if it has no sublattice isomorphic to $\mathcal{M}_5$.*

Proof.
The lattice $\mathcal{M}_5$ is not distributive. Consider the lattice from Figure 4.8. We have

$$a \vee (b \wedge c) = a \vee d$$

$$(a \vee b) \wedge (a \vee c) = e \wedge e = e$$

therefore this is not a distributive lattice.
Conversely, suppose that $\mathcal{L} = (L, \vee, \wedge)$ is a modular lattice but it is not a distributive one. There exist $x, y, z \in L$ such that

$$(x \wedge y) \vee (x \wedge z) \neq x \wedge (y \vee z) \tag{4.23}$$

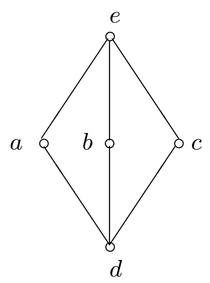


Fig. 4.8. Hasse diagram for $\mathcal{M}_5$

The dual of (4.12) is the relation

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z) \tag{4.24}$$

and by Corollary 4.1.1 the relation (4.24) is satisfied by every lattice. From (4.23) and (4.24) we deduce that the elements x, y and z satisfy the relation

$$(x \wedge y) \vee (x \wedge z) < x \wedge (y \vee z) \tag{4.25}$$

Using these elements we denote:

$$a = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

$$b = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

$$\alpha = a \vee (x \wedge b)$$

$$\beta = a \vee (y \wedge b)$$

$$\gamma = a \vee (z \wedge b)$$

Let us prove first that $a \leq b$. Taking into account the definition of a and (4.25) we obtain

$$a \leq [x \wedge (y \vee z))] \vee (y \wedge z) \tag{4.26}$$

But $\mathcal{L}$ is a modular lattice and $y \wedge z \leq y \vee z$, therefore

$$(y \wedge z) \vee [(y \vee z) \wedge x] = (y \vee z) \wedge [(y \wedge z) \vee x] \tag{4.27}$$

therefore from 4.26 we deduce

$$a \leq (y \vee z) \wedge [(y \wedge z) \vee x] \tag{4.28}$$

But $y \wedge z \leq y$ and $y \wedge z \leq z$ therefore

$$x \vee (y \wedge z) \leq x \vee y$$

$$x \vee (y \wedge z) \leq x \vee z$$

It follows that
$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) \tag{4.29}$$
From (4.28) and (4.29) we obtain
$$a \leq (y \vee z) \wedge (x \vee y) \wedge (x \vee z) = b$$
We have the following computations:
$$x \wedge b = x \wedge (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = x \wedge (y \vee z) \tag{4.30}$$
$$x \wedge a = x \wedge [(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)] \tag{4.31}$$
We observe that
$$x \vee [(x \wedge y) \vee (x \wedge z)] = x \vee (x \wedge z) = x$$
therefore $(x \wedge y) \vee (x \wedge z) \leq x$.
Applying the modular law in (4.31) we obtain
$$x \wedge a = [(x \wedge y) \vee (x \wedge z)] \wedge [x \vee (y \wedge z)] = (x \wedge y) \vee (x \wedge z)$$
because $[(x \wedge y) \vee (x \wedge z)] \leq x \leq [x \vee (y \wedge z)]$. Thus
$$x \wedge a = (x \wedge y) \vee (x \wedge z) \tag{4.32}$$
Using (4.25), (4.30) and (4.32) we obtain
$$x \wedge a < x \wedge b \tag{4.33}$$
But $a \leq b$ therefore if we suppose $a = b$ then $x \wedge a = x \wedge b$, which is not true by (4.33). Thus we proved that
$$a < b \tag{4.34}$$
We prove now the following relations:
$$\alpha \wedge \beta = \alpha \wedge \gamma = \beta \wedge \gamma = a \tag{4.35}$$
$$\alpha \vee \beta = \alpha \vee \gamma = \beta \vee \gamma = b \tag{4.36}$$
We shall prove only the relation $\alpha \wedge \beta = a$ and based on this model the reader can perform the computations for other relations.
Replacing α and β by their values we obtain:
$$\alpha \wedge \beta = [a \vee (x \wedge b)] \wedge [a \vee (y \wedge b)] = [(y \wedge b) \vee a] \wedge [a \vee (x \wedge b)]$$
But $a \leq a \vee (y \wedge b)$ therefore by the modular law we have
$$\alpha \wedge \beta = a \vee \{[(y \wedge b) \vee a] \wedge (x \wedge b)\}$$
But $a \leq b$ therefore by the modular law we have
$$(y \wedge b) \vee a = b \wedge (a \vee y)$$

therefore

$$\alpha \wedge \beta = a \vee \{[b \wedge (a \vee y)] \wedge (x \wedge b)\} = a \vee [(x \wedge b) \wedge (a \vee y)]$$

Thus we obtained the relation

$$\alpha \wedge \beta = a \vee [(x \wedge b) \wedge (a \vee y)] \tag{4.37}$$

because $(x \wedge b) \wedge b = x \wedge b$. Using the absorption law we obtain

$$x \wedge b = x \wedge (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = x \wedge (y \vee z)$$

$$a \vee y = y \vee a = y \vee (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = y \vee (x \wedge z)$$

Replacing this entities in (4.37) we obtain

$$\alpha \wedge \beta = a \vee \{x \wedge (y \vee z) \wedge [y \vee (x \wedge z)]\} \tag{4.38}$$

But $y \leq y \vee z$ therefore by modular law we have

$$(y \vee z) \wedge [y \vee (x \wedge z)] = y \vee [(y \vee z) \wedge (x \wedge z)]$$

therefore from (4.38) we obtain

$$\alpha \wedge \beta = a \vee \{x \wedge [y \vee [(y \vee z) \wedge (x \wedge z)]]\} \tag{4.39}$$

But $(y \vee z) \wedge (x \wedge z) = x \wedge z \wedge (y \vee z) = x \wedge z$ by absorption, therefore from (4.39) we obtain

$$\alpha \wedge \beta = a \vee \{x \wedge [y \vee (x \wedge z)]\} \tag{4.40}$$

Using again modular law we have

$$x \wedge [y \vee (x \wedge z)] = x \wedge [(x \wedge z) \vee y] = (x \wedge z) \vee (x \wedge y)$$

therefore from (4.40) we obtain

$$\alpha \wedge \beta = a \vee ((x \wedge z) \vee (x \wedge y)) =$$

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \vee ((x \wedge z) \vee (x \wedge y)) =$$

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = a$$

Finally let us observe that

$$a \leq a \vee (x \wedge b) = \alpha$$

$$\alpha \leq b \vee (x \wedge b) = b$$

therefore $a \leq \alpha \leq b$. similar we have $a \leq \beta \leq b$ and $a \leq \gamma \leq b$.
In conclusion, the Hasse diagram of the set $Q = \{\alpha, \beta, \gamma, a, b\}$ is represented in Figure 4.9 and we obtain a sublattice $\mathcal{Q}$ of $\mathcal{L}$ such that $\mathcal{Q}$ is isomorphic to $\mathcal{M}_5$.

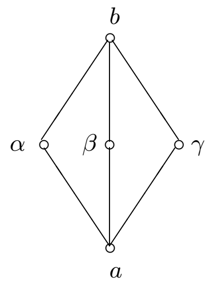


Fig. 4.9. Hasse diagram for $\mathcal{Q}$

4.2 More about Boolean algebras

4.2.1 Morphisms of Boolean algebras

We consider the Boolean algebras $\mathcal{B} = (B, \vee, \wedge, ^-, 0_B, 1_B)$ and $\mathcal{K} = (K, \Cup, \Cap, ^\star, 0_K, 1_K)$. If we consider these structures as universal algebras then a morphism $h : \mathcal{B} \longrightarrow \mathcal{K}$ is a mapping $h : B \longrightarrow K$ that satisfies the following conditions:

$$h(x \vee y) = h(x) \Cup h(y) \tag{4.41}$$

$$h(x \wedge y) = h(x) \Cap h(y) \tag{4.42}$$

$$h(\overline{x}) = (h(x))^\star \tag{4.43}$$

$$h(0_B) = 0_K \tag{4.44}$$

$$h(1_B) = 1_K \tag{4.45}$$

In this section we show that not all these identities are independent and thus we obtain a short characterization for a morphism of Boolean algebras.

Proposition 4.2.1. *We have the following equivalences:*

$$\{(4.41), (4.42), (4.43), (4.44), (4.45)\} \Longleftrightarrow \{(4.41), (4.43)\}$$

$$\{(4.41), (4.42), (4.43), (4.44), (4.45)\} \Longleftrightarrow \{(4.42), (4.43)\}$$

$$\{(4.41), (4.42), (4.43), (4.44), (4.45)\} \Longleftrightarrow \{(4.41), (4.42), (4.44), (4.45)\}$$

Proof. We have the following implications:
1) $(4.41), (4.43) \Longrightarrow (4.42)$:

$$h(x \wedge y) = h(\overline{\overline{x \wedge y}}) = (h(\overline{x \wedge y}))^\star = (h(\overline{x} \vee \overline{y}))^\star = (h(\overline{x}) \Cup h(\overline{y}))^\star =$$

$$((h(x))^\star \Cup (h(y))^\star)^\star = ((h(x))^{\star\star} \Cap (h(y))^{\star\star}) = h(x) \Cap h(y)$$

2) $(4.41), (4.43) \Longrightarrow (4.44)$:

$$h(0_B) = h(x \wedge \overline{x}) = h(x) \Cap h(\overline{x}) = h(x) \Cap (h(x))^\star = 0_K$$

3) $(4.41), (4.43) \Longrightarrow (4.45)$:

$$h(1_B) = h(x \vee \overline{x}) = h(x) \Cup h(\overline{x}) = h(x) \Cup (h(x))^\star = 1_K$$

Similar we prove the other implications. ∎

4.2.2 Stone theorem (finite case)

In this section we present a fruitful result known as *Stone theorem*. The result is proved only for the particular case of the finite sets. Various applications use this result.

Definition 4.2.1. *Consider a Boolean algebra $\mathcal{B} = (B, \vee, \wedge, ^-, 0, 1)$. An* **atom** *of $\mathcal{B}$ is an element $a \in B$ such that $a \neq 0$ and for every $x \in B$ from $0 \leq x \leq a$ we deduce $x = 0$ or $x = a$.*

Remark 4.2.1. Various interesting properties of a Boolean algebra can be characterized by means of this concept. For example, $B = \{0, 1\}$ *if and only if* 1 *is an atom of* B. Really, if 1 is an atom then by Definition 4.2.1 for every $x \in B$ we have $x = 0$ or $x = 1$ therefore $B = \{0, 1\}$. The converse implication is immediately obtained by the same definition: if by contrary, 1 is not an atom then there is $x \in B$ such that $0 < x < 1$, therefore B contains at least three elements.

A trivial example of Boolean algebra is

$$\mathcal{M} = (2^M, \cup, \cap, C, \emptyset, M)$$

where $\cup$ and $\cap$ are set theoretical union and intersection and $C(A) = M \setminus A$. Let us characterize its atoms.

Proposition 4.2.2. *The element $A \in 2^M$ is an atom of $\mathcal{M}$ if and only if A is a singleton.*

Proof. If A is an atom then $A \neq \emptyset$ and for every $K \in 2^M$ from $\emptyset \subseteq K \subseteq A$ we deduce $K = \emptyset$ or $K = A$. If by contrary A is not a singleton then consider $b \in A$, take $K = \{b\}$ and we have $\emptyset \subset K \subset A$, therefore A is not an atom of $\mathcal{M}$. The converse implication is obviously true. The Boolean algebra $\mathcal{M} = (2^M, \cup, \cap, C, \emptyset, M)$ is named the *Boolean algebra of the power set of M*. ∎

Proposition 4.2.3. *In every finite Boolean algebra $\mathcal{B} = (B, \vee, \wedge, ^-, 0, 1)$ the following property is satisfied: for every $x \in B \setminus \{0\}$ there exists an atom $a \in B$ such that $a \leq x$.*

Proof. Suppose $x \in B \setminus \{0\}$. Denote $y_1 = x$. Either y_1 is an atom and in this case the property is verified, or there is $y_2 \in B$ such that $0 < y_2 < y_1$. Now we repeat the previous reasoning. Either y_2 is an atom or there is $y_3 \in B$ such that $0 < y_3 < y_2 < y_1$. This process is finite one because B is a finite set. ∎

Proposition 4.2.4. *If x and y are atoms and $x \neq y$ then $x \wedge y = 0$.*

Proof. The element y is an atom and $0 \leq x \wedge \leq y$ therefore either $x \wedge y = 0$ or $x \wedge y = y$. If $x \wedge y = 0$ then the property is satisfied. Otherwise $x < y$ because from $x \wedge y = y$ we deduce $x \leq y$. Thus we have $0 < x < y$ and this relation is not possible because y is an atom. ∎

Proposition 4.2.5. *If $\mathcal{B} = (B, \vee, \wedge, ^-, 0, 1)$ is a finite Boolean algebra and A is the set of all its atoms then for every $x \in B \setminus \{0\}$ we have*

$$x = \bigcup_{a \in A, a \leq x} a \tag{4.46}$$

Proof. We denote

$$A_x = \{a \in A \mid a \leq x\}$$

and we prove that $x = supA_x$.

Directly from the definition of A_x we observe that x is an upper bound for A_x. Suppose that z is also an upper bound for the same set. We verify first that $x \leq z$. By contrary we suppose $x \not\leq z$. Equivalently we have $x \cdot \overline{z} \neq 0$. Applying Proposition 4.2.3 it follows that there is an atom $a_0 \in A$ such that $a_0 \leq x \cdot \overline{z}$. But $x \cdot \overline{z} \leq x$ therefore $a_0 \leq x$. In other words, $a_0 \in A_x$. The element z is an upper bound for A_x therefore $a_0 \leq z$. From the inequality $a_0 \leq x \cdot \overline{z}$ we deduce $a_0 \leq \overline{z}$ and if we combine this relation with $a_0 \leq z$ then we obtain $a_0 = 0$, which is not possible because a_0 is an atom. Thus the assumption $x \not\leq z$ is false, therefore $x \leq z$ and the proposition is proved. ∎

Proposition 4.2.6. *(Stone theorem) Every finite Boolean algebra is isomorphic to the Boolean algebra of the power set of its atoms.*

Proof. Consider a finite Boolean algebra $\mathcal{B} = (B, \vee, \wedge, ^-, 0, 1)$ and denote by A the set of its atoms. By Proposition 4.2.3 we have $A \neq \emptyset$. We define the mapping $h : B \longrightarrow 2^A$ by

$$h(x) = \{a \in A \mid a \leq x\}$$

The following properties are satisfied by h:

1) *The mapping h is surjective.*

Consider $Y \subseteq A$. Denote $x = supY$. Obviously $Y \subseteq h(x)$ because if $y \in Y$ then $y \in A$ and $y \leq x$. Conversely, consider $a_1 \in h(x)$ and let us verify that $a_1 \in Y$. By contrary we suppose $a_1 \notin Y$. For every $y \in Y$ we have $0 \leq y \wedge a_1 \leq a_1$. But a_1 is an atom therefore by Proposition 4.2.4 we have $y \wedge a_1 = 0$. It follows that $y \leq \overline{a}_1$. This inequality is satisfied for every $y \in Y$ therefore $\overline{a}_1$ is an upper bound for Y. But $x = supY$ therefore $x \leq \overline{a}_1$ and thus $x \wedge a_1 = 0$. We observe that $a_1 \leq x$ because $a_1 \in h(x)$. It follows that $a_1 \wedge x = a_1$ and thus $a_1 = 0$, which is not possible because a_1 is an atom.

2) *The mapping h is a morphism of Boolean algebras.*
The condition

$$h(x \wedge y) = h(x) \cap h(y) \tag{4.47}$$

is verified because the following conditions are equivalent:

$a \in h(x \wedge y)$
$a \in A$ and $a \leq x \wedge y$
$a \in A$, $a \leq x$ and $a \leq y$
$a \in h(x) \cap h(y)$

The following condition is also verified:

$$h(\overline{x}) = C(h(x)) \tag{4.48}$$

Really, if $a \in h(\overline{x})$ then $a \in A$ and $a \leq \overline{x}$. If by contrary we suppose that $a \in h(x)$ then $a \leq x$ and therefore $a \leq x \wedge \overline{x} = 0$, which is not possible because $a \in A$.
Conversely, suppose $a \in C(h(x))$. It follows that $a \in A \setminus h(x)$ because $C(h(x)) = A \setminus h(x)$. We observe that $0 \leq a \wedge x \leq a$ and $a \in A$. Suppose $a \wedge x = a$, therefore $a \leq x$. In this case $a \in h(x)$, which is not possible. It remains that $a \wedge x = 0$, therefore $a \leq \overline{x}$. Thus $a \in h(\overline{x})$.

3) *The mapping h is injective.*
Suppose $h(x) = h(y)$. In other words,

$$\{a \in A \mid a \leq x\} = \{a \in A \mid a \leq y\} \tag{4.49}$$

Using (4.46) and (4.49) we obtain

$$x = \bigcup_{a \in A, a \leq x} a = \bigcup_{a \in A, a \leq y} a = y$$

and the proposition is proved. ■

Corollary 4.2.1. *If $\mathcal{B}$ is a finite Boolean algebra containing n atoms then $\mathcal{B}$ has 2^n elements.*

Proof. By Stone theorem The Boolean algebra $\mathcal{B}$ is isomorphic to the Boolean algebra $(2^A, \cup, \cap, \emptyset, A)$, where A is the set of all atoms of $\mathcal{B}$. If the cardinal of A is n then the cardinal of the set 2^A is 2^n. ■

Remark 4.2.2. The structure represented in Figure 4.4 is a Boolean algebra and the set of its atoms is $A = \{2, 3, 5\}$. We observe that this structure contains $2^3 = 8$ elements. The mapping h from Proposition 4.2.6 is the following:

$$h(0) = \emptyset; h(2) = \{2\}; h(3) = \{3\}; h(5) = \{5\}$$

$$h(6) = \{2, 3\}; h(10) = \{2, 5\}; h(15) = \{3, 5\}; h(30) = \{2, 3, 5\}$$

Moreover, we have the following computations:

$$h(\overline{2}) = C(h(2)) = A \setminus \{2\} = \{3, 5\} = h(15)$$

therefore $\overline{2} = 15$;

$$h(\overline{3}) = C(h(3)) = A \setminus \{3\} = \{2, 5\} = h(10)$$

therefore $\overline{3} = 10$.
Similar we obtain $\overline{5} = \{6\}$, $\overline{6} = \{5\}$, $\overline{10} = \{3\}$, $\overline{15} = \{2\}$, $\overline{0} = \{30\}$ and $\overline{30} = \emptyset$.

Remark 4.2.3. Various "Stone concepts" can be encountered in literature: Stone lattice, Stone space, Stone isomorphism, Stone algebra, Stone duality etc. We observe that only the finite case for the Stone theorem is treated in this section. The general case is treated by reference books as Rasiowa and Sikorski (1963), Grätzer (1971), Grätzer (1978).

5. Connections and perspectives

Several implications of the lattices in computer science can be relieved. In this section we present some aspects which can invite the reader to find possible subjects for their research.

First of all we relieve the fact that the treatment of this volume is not an exhaustive one. In preparing this volume we taken into account the following aspects:

- we intended to offer a self contained volume for those readers which are interested to study the use of universal algebra in knowledge representation by inheritance, labeled stratified graphs and semantic schemas; the reader can find various references to these concepts and methods in the last part of this section and also in the **References** part of this volume;
- we intended to introduce the reader into some algebraic domain with large perspectives both in applied mathematics and theoretical computer science;
- the persons interested in this domain can themselves to accomplish a supplementary documentation to find a proper research line, to develop theoretical results or to discover practical applications.

1. An excellent book in the domain of lattice theory is Rudeanu (2001), which is strongly connected by the book Rudeanu (1974). Several applications of lattices and Boolean algebras are described in the following domains: graph theory, automata theory, synthesis of circuits, fault detection in combinational circuits, marketing, databases, numerical analysis.
2. The concept of lattice is strongly implied at the frontier of mathematics and computer science.

   • A first implication we relieve here is in the *theory of confidence*, which enables us to attribute degrees of confidence to propositions. The degrees of confidence are characterized by the elements of an appropriate lattice with first and last elements. They are attributed to the formulas of a language when the propositional variables are interpreted as denoting specific vague statements. In order to do so, let $(L, \sqcup, \sqcap, 0, 1)$ be a lattice with first and last elements. As usual we denote by $\leq$ the partial order of L. If F is the set of formulas let $C : F \longrightarrow L$ be a mapping satisfying the following conditions, where I is a finite set of indices:

$$C(\alpha \wedge \neg\alpha) = 0$$

$$C(\alpha \vee \neg\alpha) = 1$$

$$C(\vee_{i\in I}\alpha_i) \geq \sqcup_{i\in I} C(\alpha_i)$$

$$C(\wedge_{i\in I}\alpha_i) \leq \sqcap_{i\in I} C(\alpha_i)$$

$$If \ \vdash \alpha \leftrightarrow \beta \ then \ C(\alpha) = C(\beta)$$

Such a mapping C is called a *confidence function*. Depending on the applications, the above conditions can be extended. For example we can introduce the following condition:

$$If \ \vdash \alpha \rightarrow \beta \ then \ C(\alpha) \leq C(\beta)$$

The reader can find in Costa and Krause (2002) a proposal for an *algebra of confidence* with large implication in the study of vagueness and other kinds of propositional logic (propositions are vague in some sense but, despite their vagueness, they can be believed with a certain degree of confidence). The reader can find rich ideas if follows the research line of da Costa.

• A second implication of the lattice theory can be encountered in Ginsberg (1986), where the structure of *bilattice* is presented and the applications in logic programming are discussed. A bilattice is a structure $(B, \leq_t, \leq_k, \neg)$ consisting of a non-empty set B, two partial orderings $\leq_t$ and $\leq_k$ on B and a mapping $\neg : B \longrightarrow B$ such that:

$(B, \leq_t)$ and $(B, \leq_k)$ are complete lattices;
$x \leq_t y \Longrightarrow \neg y \leq_t \neg x$;
$x \leq_k y \Longrightarrow \neg x \leq_k \neg y$;
$\neg\neg x = x$

The negation operator establishes the connection between the two orderings. The reader interested in the domain of logic programming can find an interesting study of the distributive bilattices and the fixpoint semantics using bilattices (and interlaced bilattices) in Fitting (1990) and Fitting (1991).

• A third major implication of the lattice theory is connected by fuzzy theory. We only underline here this fruitful research line because there is a great number of papers and books related of this subject (Ajmal and Thomas (1994), Kehagias (2004) etc).

3. Various applications of Boolean functions can be relieved. We intend only to present this concept here and to establish a possible research line.
   In a Boolean algebra $(B, \vee, \wedge, ^{-}, 0, 1)$ we denote $x^0 = \overline{x}$ and $x^1 = x$. A *Boolean function* of n variable is a mapping $f : B^n \longrightarrow B$ such that for every $x_1, \ldots, x_n \in B$ we have

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{a_1,\ldots,a_n \in \{0,1\}} f(a_1, \ldots, a_n) x_1^{a_1} \ldots x_n^{a_n} \qquad (5.1)$$

   Not every mapping can be written as (5.1). For example, take $B = \{0, 1, a, \overline{a}\}$ and the mapping of one variable $f : B \longrightarrow B$ defined by $f(a) = 1$ and $f(0) = f(1) = f(\overline{a}) = 0$. If we suppose that this is a Boolean function then $f(x) = f(0) \cdot \overline{x} \vee f(1) \cdot x = 0$ for every $x \in B$ and this property is not true. Thus f is not a Boolean function. Various aspects related to Boolean functions and equations are treated in Rudeanu (1974).
4. The study of non Boolean functions defined on Boolean algebras can open various perspectives in the research work. A possible research line in this domain is in connection with the concept of *generalized Boolean function* introduced in Ţăndăreanu (1981) and developed in a subsequent papers.

   We consider a finite set $A = \{a_1, \ldots, a_n\}$ such that $\{0, 1\} \subseteq A \subset B$, where B is a Boolean algebra. We denote by $G(A)$ the set of all functions

$$g : A \times B \longrightarrow B$$

   such that

$$g(0, 0) = g(1, 1) = 1$$

   and for every $x \in B$ the set $\{g(a_1, x), \ldots, g(a_n, x)\}$ is orthonormal:

$$g(a_1, x) \vee \ldots \vee g(a_n, x) = 1$$

$$g(a_i, x) \cdot g(a_j, x) = 0, \; i \neq j$$

   If $g \in G(A)$ then the mapping $f : B^n \longrightarrow B$ is a *g-generalized Boolean function* if it satisfies the following identity:

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{a_1,\ldots,a_n \in A} f(a_1, \ldots, a_n) g(a_1, x_1) \ldots g(a_n, x_n)$$

   A mapping $f : B^n \longrightarrow B$ is an A-generalized Boolean function if there is $g \in G(A)$ such that f is a g-generalized Boolean function.

   If we denote by $BF_n(B)$ the set of all Boolean functions of n variables defined on B and by $GBF_n(A, B)$ the set of all A-generalized Boolean functions then

$$BF_n(B) \subset GBF_n(A, B) \subset B^{B^n} \qquad (5.2)$$

   The monotonicity of the generalized Boolean functions of one variable is studied in Ţăndăreanu (1985b) and the concept of partial derivative is studied in Ţăndăreanu (1985a).

5. The following research directions can be developed:
   - Based on (5.2) try to approximate a generalized Boolean function by means of Boolean functions.
   - Based on the concept of interval in a Boolean algebra study those generalized Boolean functions that are Boolean function on some intervals.
   - Identify classes of generalized Boolean functions satisfying various restrictions (isotone on some intervals and antitone on other intervals etc).
   - The Boolean functions were successfully applied to combinational circuits. Identify a corresponding application for generalized Boolean functions.
   - The Boolean functions were applied to solve Boolean equations. Introduce the concept of generalized Boolean equation and find methods to solve these equations. Extend this problem to systems of such equations. Find problems that can be modeled by such equations.
6. The reader interested to accomplish a detailed study concerning the implications of these structures in logic (category theory, Heyting algebras) can use Buşneag (1997). An interesting presentation of the connections between universal algebras and a branch of logic called model theory can be found in Burris (1981).
7. The following arguments can be used to argue the interest in the structures presented in this volume:
   - The implications of the lattice theory into computability of the answer mapping for inheritance based knowledge systems can be observed in the papers Ţăndăreanu (1999), Ţăndăreanu (2001b), Ţăndăreanu (2002a), Ţăndăreanu (2003d) as vwell as in the book Ţăndăreanu (2004e).
   - The concepts of Peano algebra, morphisms of partial algebras and semilattices are deeply used to introduce two mechanisms for knowledge representation: *labeled stratified graphs* (Ţăndăreanu (2004b) etc) and *semantic schemas* (Ţăndăreanu (2004c) etc)

References

Ajmal N.and Thomas K.V. (1994): Fuzzy lattices, Info. Sciences. vol. 79, pp.271-291

Boicescu V., Filipoiu A., Georgescu G. and Rudeanu S. (1991): "Lukasiewicz-Moisil Algebra". *Annals of Discrete Mathematics* **49** North-Holland

Burmeister P. (2002): Lecture Notes on Universal Algebra. Many-Sorted Partial Algebras,Darmstadt university of Technology, http://www.mathematik.tu-darmstadt.de/~burmeister/

Burris S., Sankappanavar H.P. (1981): A Course in Universal Algebra, Graduate Texts in Mathematics No. 78, Springer-Verlag (On-Line Millennium Edition at http://www.math.uwaterloo.ca/ snburris/htdocs/MYWORKS/publ.html)

Buşneag D. (1997): Special chapters of algebra, Universitaria Press, Craiova (in Romania)

Căzănescu V.E. (1975): Course of Computer Science Fundamentals, Bucharest University, Lesson I, University Publishing Center (in Romanian)

da Costa, N. C. A. and Krause, D., An inductive annotated logic, in W. Carnielli, M. Cogniglio and I. M. L. D'Ottaviano (eds.), Paraconsistency: the Logical Way to the Inconsistent, Proceedings do II World Congress on Paraconsistency, Juquehy, SP, May 2000 (marcel Dekker, New York), 213-225

M. Fitting (1990): Bilattices in logic programming. The 20th Int. Symp. on Multiple-Valued Logic, IEEE Press, 1990

M. Fitting (1991): Bilattices and the semantics of logic programming. Journal of Logic Programming, 11(1–2):91-116

M. L. Ginsberg, Multi-valued logics, Proc. AAAI-86, fifth national conference on artificial intelligence, pp 243-247, Morgan Kaufmann Publishers

Grätzer G. (1971): Lattice Theory. First Concepts and Distributive Lattices, W.H.Freeman and Co., San Francisco

Grätzer G. (1978): General Lattice Theory, Akademie-Verlag, Berlin.

Ljapin E.S., Evseev A.E. (1997): The Theory of Partial Algebraic Operations, Kluwer Academic Publishers, Mathematics and Its Applications Volume 414

Newton C. A. da Costa, Decio Krause (2000): An inductive annotated logic,

Kehagias (2004): The Lattice of Fuzzy Intervals and Sufficient Conditions for Its Distributivity, arXiv:cs.OH/0206025 v1

Rasiowa,H., Sikorski, R.(1963): The Mathematics of Metamathematics, Panstwowe Wydawnictwo Naukowe Warszawa (Poland), Tom 41

Rudeanu S. (1974): Boolean functions and equations, North-Holland, Amsterdam.

Rudeanu S. (2001): Lattice Functions and Equations, Discrete Mathematics and Computer Science, Springer

Ţăndăreanu, N. (1981): On generalized Boolean functions I, Discrete Math. 34, 293-299

Ţăndăreanu, N. (1982): On generalized Boolean functions II, Discrete Math. 40, 277-284

Ţăndăreanu, N. (1984a): On generalized Boolean functions III. The case $A = \{0, 1\}$, Discrete Math. 52, 269-277

Ţăndăreanu, N. (1984b): Remarks on generalized Boolean functions of one variable. Rev. Roumanie Math. Pures Appl. 29, 715-719

Ţăndăreanu, N. (1985a): Partial derivatives of $\{0, 1\}$-generalized Boolean functions. An. Univ. Craiova, Ser. mat.-Fiz.-Chim. 13, 63-66

Ţăndăreanu, N. (1985b): Monotonicity of the functions of one variable defined on Boolean algebras. Rev. Roumanie Math. Pures Appl. 30, 579-582

Ţăndăreanu, N. (1999): Lattices in Knowledge Representation by Frames, Proceedings of the 3th Annual Meeting of The Romanian Society of Mathematical Sciences- Computer Science Series, 26-29 May, Craiova, Vol.2, 1999, p.10-19.

Ţăndăreanu, N. (2000a): Proving the Existence of labeled Stratified Graphs, *Annals of the University of Craiova*, Vol. XXVII, 81–92; on line version at http://inf.ucv.ro/~ntand/en/publications.html

Ţăndăreanu, N. (2001b): Lattices of Labelled Ordered Trees (I), Annals of the University of Craiova, Mathematics and Computer Science Series, Vol. XXVIII, 2001, 29-39

Ţăndăreanu, N. (2002a): Lattices of Labelled Ordered Trees (II), Annals of the University of Craiova, Mathematics and Computer Science Series, Vol. XXIX, 2002, 137-144

Ţăndăreanu, N. (2003d): Inheritance-based knowledge systems and their answer functions computation using lattice theory, Romanian Journal of Information Science and Technology, Vol.6, Numbers 1-2, 2003, 227-248

Ţăndăreanu, N. (2004b): Distinguished Representatives for Equivalent labeled Stratified Graphs and Applications, Discrete Applied Mathematics, Vol. 144/1-2, p.183-208

Ţăndăreanu, N. (2004c): Knowledge Representation by Labeled Stratified Graphs, The 8th World Multi-Conference on Systemics, Cybernetics and Informatics, July 18-21, Orlando, Florida, Vol. V: Computer Science and Engineering, p.345-350

Ţăndăreanu, N. (2004d): Semantic Schemas and Applications in Logical representation of Knowledge, Proceedings of the 10th International Conference on Cybernetics and Information Technologies, Systems and Applications (CITSA2004), July 21-25, Orlando, Florida, Vol. III p.82-87

Ţăndăreanu, N. (2004e): Knowledge Bases, Sitech Press, ISBN 973-657-720-1 (in Romanian)

Ţăndăreanu, N. (2005a): Transfer of knowledge via semantic schemas, July 10-13, Vol. IV, p.70-75, 2005

Ţăndăreanu, N., Ghindeanu, M. (2005b): A three-level distributed knowledge system based on semantic schemas, 16th Int. Workshop on Database and Expert Systems Applications, Proceedings of DEXA'05, Copenhagen, p.423-427, 2005

Ţăndăreanu N., Ghindeanu M. (2006)- Properties of derivations in a Semantic Schema, Annals of University of Craiova, Math. Comp. Sci. Ser. (to appear)

Ţăndăreanu, N. (2006a): The Greatest Lower Bound of Two Semantic Schemas, 3rd International Conference on Cybernetics and Information Technologies, Systems and Applications (CITSA 2006), USA, July 20-23, 2006 (to appear).

Ţăndăreanu, N. (2006b): The Use of Structured Elements to Obtain Interpretations for Semantic Schemas, 10th World Multiconference on Systemics, Cybernetics and Informatics (WMSCI 2006), USA, July 16-19, 2006 (to appear).

Ţăndăreanu, N. (2006c): Semantic Schemas: The Least Upper Bound of Two Interpretations, 10th World Multiconference on Systemics, Cybernetics and Informatics (WMSCI 2006), USA, 16-19, 2006 (to appear).