Towards a Mathematical Modelling of Conditional Knowledge

Nicolae Ţăndăreanu[1] and Mihaela Ghindeanu[2]
Faculty of Mathematics and Computer Science,
Department of Computer Science,
University of Craiova, Romania

[1] ntand@oltenia.ro
[2] mghindeanu@yahoo.com

Abstract. In this paper we propose a graph-based formalism to represent conditional knowledge, that is, the knowledge pieces containing sentences of the form *if-then*. We introduce the concept of *conditional scheme*, which is a knowledge representation and reasoning system based on this formalism. The corresponding reasoning formalism is introduced and the answer function for such a system is defined. All the concepts are exemplified on a particular case of a given knowledge piece.
Keywords: knowledge representation, directed graph, conditional scheme
AMS Subject Classification: 68T30, 68T35

1 Introduction

There are two main kinds of methods to represent the knowledge: graph-based methods and logical methods.

Based on the graph theory, a great number of research works deal with the developing of methods for knowledge representation. A very productive notion with large applications in knowledge representation, is that of *conceptual graph*, a notion introduced in literature by J.W.Sowa ([1],[2], [3], [4], [5]). Other works treat the concepts of *labelled stratified graph* and *knowledge base with output* ([6], [7]). A great number of models do not treat the representation of the knowledge pieces containing sentences of the form *if - then*, which will be named in this paper *conditional knowledge*. On the other hand, the negative answer to an interrogation and the corresponding explanations are difficult problems in the representation based on graph theory. These problems are treated, in general, in models based on logic representation. A possible approach of this problem is treated in the present paper.

In this paper we try to introduce a graph-based formalism that allows us to represent conditional knowledge. The formalism to process the knowledge represented by this method is introduced, particularly the answer function for a system using this method is defined. In order to relieve the intuitive aspects, all the concepts are exemplified.

The paper is organized as follows: in Section 2 several preliminary notions are introduced; in Section 3 an example of knowledge piece containing conditional entities is considered; in Section 4 the concept of *conditional scheme* is introduced and the reasoning formalism for such scheme is defined; the computations are exemplified for the particular case of the example taken in Section 3. It is relieved also the problem of a negative answer to an interrogation applied to a conditional scheme as well as the capabilities of such scheme to give the corresponding explanations in this case. Several future works are presented in the last section.

2 Intuitive aspects

In this section we present the intuitive meaning for several aspects that are formalized in the next section.

In general, from a knowledge piece KP some directed graph is obtained. The nodes of this graph are the objects of KP and the arcs are given by the binary relations specified in KP. The originality of our method consists in the usage of the *conditional binary relations*. A binary relation ρ on the set X is a subset $\rho \subseteq X \times X$ and a binary relation from X to Y is a subset of $X \times Y$. Thus, if Peter is John's brother and Mike is George's brother then the binary relation $is\_brother = \{(Peter, John), (Mike, George)\}$ is obtained. Let us now consider the following sentences:

If Bob lives in a fish bowl then it is a fish.
If Peter obtains a good score at the school then he is a competitor.

In order to describe this situation we consider the systems:

$((Bob, fish), p_1)$
$((Peter, competitor), p_2)$

where p_1 is the condition *Bob lives in a fish bowl* and p_2 represents the condition *Peter obtains a good score*. In this way for the set

$$X = \{Bob, fish, Peter, competitor\}$$

we obtained the binary relation

$$\rho = \{(Bob, fish), (Peter, competitor)\}$$

and if we denote $P = \{p_1, p_2\}$ then the folowing relation from $X \times X$ to P is obtained:

$$\theta = \{((Bob, fish), p_1), ((Peter, competitor), p_2)\}$$

We shall say that θ is a *conditional binary relation*. In the intuitive meaning we have:

$(Bob, fish)$ belongs to ρ
$(Bob, fish)$ belongs to θ if p_1 is true

In other word, by the conditional relation θ we represent the following sentences:

If p_1 then $(Bob, fish) \in \theta$
If p_2 then $(Peter, competitor) \in \theta$

Every binary relation can be considered a conditional binary relation. Really, the above relation ρ can be written:

$$\rho = \{((Bob, fish), T), ((Peter, competitor), T)\}$$

where $T = true$.

Frequently the conditions p_1 and p_2 are expressed in terms of *initial knowledge* about the used objects. The initial knowledge for some object will be specified in the following form:

$$(attribute\_name, value)$$

that is, an attribute for the corresponding object and its value.

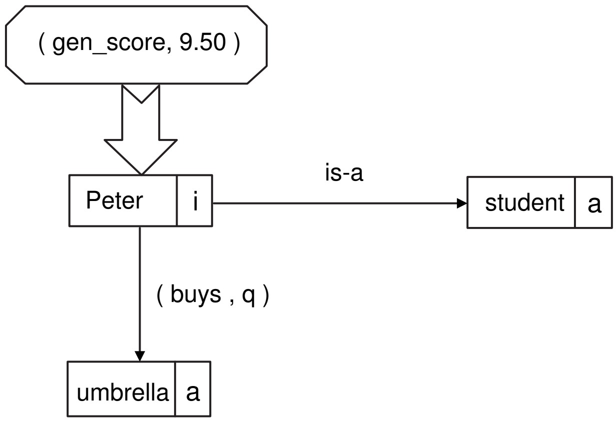

Fig. 1. Graphical representation of $KP1$

For example, let us consider the following knowledge piece $KP1$:

Peter is a student. His general score at university is 9.50. Peter buys an umbrella if it is rain.

We remark that Peter is *some* student. He is an *individual* object. The object *student*, as well as *umbrella*, is an *abstract* object. Graphically this will be represented as in Figure 1, where the characters i and a specify an individual or and abstract object, respectively. In order to specify a conditional binary relation we shall write above the corresponding arc of the graph the pair $(rel, cond)$, where rel designates the name of the relation and $cond$ represents the symbol for the condition implied by the objects. From $KP1$ we extract:

- the classical binary relation $is\_a = \{(Peter, student)\}$; equivalently we can consider the conditional binary relation $is\_a = \{((Peter, student), T)\}$

- the conditional binary relation $buys = \{((Peter, umbrella), q)\}$, where q represents the condition *it is rain*.

3 The decomposition of a knowledge piece.

In this section we shall consider an example of knowledge piece and we extract and decompose its knowledge into several entities according to the formalism defined in the previous section. More precisely, the following knowledge piece $KP2$ is considered:

Peter, George, Mike and Alin are students. Peter is George's friend. George is Mike's brother and Peter is Alin's brother. Every student plays basketball if he is tall. Peter is tall. The basketball is a team sport. Every student participates to the final competition if his general score at university is greater than 9. Every competitor obtains a mention if $8 < (score1 + score2)/2 \leq 9$*, where score1 and score2 represent the results obtained in competition. Every competitor obtains a special prize if* $(score1 + score2)/2 > 9$*. The following general scores was realized at university:*

Peter: 9.50; George: 9.20
Mike: 9.60; Alin: 6

The scores obtained in the competition are the following:

Peter: score1=8; score2=9;
George: score1=8; score2=10;
Mike: score1=9

From the above text we extract the following objects:

- Individual objects: Peter, George, Mike, Alin
- Abstract objects: student, basketball, competitor, team sport, mention, prize

We consider the following attributes for objects:

- gen_score to represent the general score obtained at university by a student
- $score1$ and $score2$ to represent the results obtained in the competition by some student
- $height$ to represent the height of a person

In order to avoid a possible confusion we shall use the following notation

$$a \Longleftrightarrow \{(x, y), \ldots, (u, v)\}$$

to specify that the binary relation $\{(x, y), \ldots, (u, v)\}$ is represented by the symbol a.

The following binary relations are extracted from the text of $KP2$:

- $friend\_of \Longleftrightarrow \{(Peter, George)\}$
- $brother\_of \Longleftrightarrow \{(George, Mike), (Peter, Alin)\}$

We extract also the following conditional binary relations:

- $plays \Longleftrightarrow \{((student, basketball), s)\}$

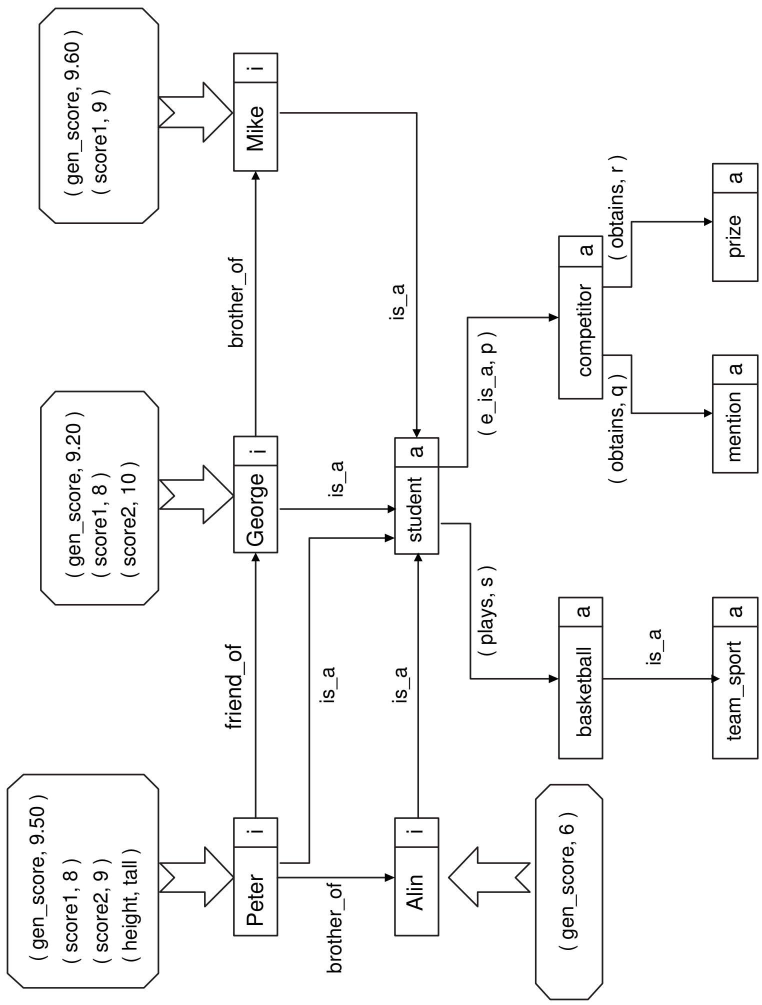

Fig. 2. Graphical representation of $KP2$

- $is\_a \Longleftrightarrow \{((George, student), T), ((Peter, student), T),$
  $((Alin, student), T), ((Mike, student), T), ((basketball, team\_sport), T)$
- $e\_is\_a \Longleftrightarrow \{((student, competitor), p)\}$
- $obtains \Longleftrightarrow \{((competitor, mention), q), ((competitor, prize), r)\}$

In the above description we have two similar conditional binary relations, namely is_a and e_is_a. However, there is a semantical difference between them and this will be explained later.

For an object x, if $(attr, value)$ is attached to x then we shall denote

$$V_x(attr) = value$$

in order to specify that *value* is refered to x. Using these notations for $KP2$ we obtain:

- $V_{Peter}(gen\_score) = 9.50$; $V_{Peter}(score1) = 8$;
- $V_{Peter}(score2) = 9$; $V_{Peter}(height) = tall$;
- $V_{George}(gen\_score) = 9.20$; $V_{George}(score1) = 8$; $V_{George}(score2) = 10$;
- $V_{Mike}(gen\_score) = 9.60$; $V_{Mike}(score1) = 9$; $V_{Alin}(gen\_score) = 6$;

The conditions are denoted by the symbols p, q, r, s. They are named *conditional symbols* and we denote $\mathcal{C}_{sym} = \{p, q, r, s\}$. Some logical condition is attached to every element of $\mathcal{C}_{sym}$.

Intuitively, for each symbol $t \in \mathcal{C}_{sym}$ and object x we say that *t is on* for x if the condition attached is satisfied for x and *t is off* otherwise. We shall write $t(x) = on$ or $t(x) = off$.

In order to describe the computation of these values we shall use the method given by the rules IF-THEN. For $KP2$ we obtain the following rules:

- $R_1(x)$: IF $V_x(gen\_score) > 9$ THEN $p(x) = on$ ELSE $p(x) = off$
- $R_2(x)$: IF $8 < (V_x(score1) + V_x(score2))/2 \leq 9$ THEN $q(x) = on$ ELSE $q(x) = off$
- $R_3(x)$: IF $(V_x(score1) + V_x(score2))/2 > 9$ THEN $r(x) = on$ ELSE $r(x) = off$
- $R_4(x)$: IF $V_x(height) = tall$ THEN $s(x) = on$ ELSE $s(x) = off$

where R_j is the name of a rule and $R_j(x)$ denotes the property that the rule R_j is applied to the object x.

Using this formalism we obtain the representation given in Figure 2. Now, the intuitive aspects are presented. It remains to formalize all theses aspects and to obtain a knowledge representation and reasoning system based on this formalism. This is treated in the next section.

4 Conditional scheme and reasoning formalism

For a given scheme we shall use the following notations:

- *Ob*, the set of all objects extracted from a given knowledge piece

- $\mathcal{C}_{sym}$, the set of conditional symbols; for $KP2$ we have $\mathcal{C}_{sym} = \{p, q, r, s\}$
- E, the set of the symbols for binary/conditional binary relations; for $KP2$ we have $E = \{friend\_of, brother\_of, is\_a, e\_is\_a, plays, obtains\}$
- A, the set of all attribute names
- $I = \{i, a\}$, where i is used to designate an individual object and a for abstract object.

Definition 1. *Let $G = (X, \Gamma)$ be a directed graph, where $X \subseteq Ob \times I$ is the set of nodes; an element of Γ is named* **arc** *and it is of the form*

$$((n, k_1), (a, p), (m, k_2))$$

where $(n, k_1), (m, k_2) \in X$ and $(a, p) \in E \times (\mathcal{C}_{sym} \cup \{T\})$. G is called a **directed graph with conditional relations**.

For example, in the graph of $KP2$ we have

$$((student, a), (plays, s), (basketball, a)) \in \Gamma$$

$$((Peter, i), (is\_a, T), (George, i)) \in \Gamma$$

and so on.

Definition 2. *Let be $n_1, n_{k+1} \in Ob$ two arbitrary objects. A path from n_1 to n_{k+1} in G is a pair of the form*

$$([(n_1, w_1), \ldots, (n_{k+1}, w_{k+1})], [(a_1, t_1), \ldots, (a_k, t_k)])$$

such that the following conditions are fulfilled:

- $(n_j, w_j) \in X,\ j \in \{1, \ldots, k+1\}$
- $(a_j, t_j) \in E \times (\mathcal{C}_{sym} \cup \{T\}),\ j \in \{1, \ldots, k\}$
- $((n_j, w_j), (a_j, t_j), (n_{j+1}, w_{j+1})) \in \Gamma,\ j \in \{1, \ldots, k\}$

We denote by $Path(n_1, n_{k+1})$ the set of all paths from n_1 to n_{k+1} in G.

We shall observe that a binary relation is composed from all the pairs of objects that are related by the same property. For example, from the sentences

Peter is a student
Basketball is a team sport
Bob is a child if he drinks milk

we obtain the conditional binary relation

$$\{((Peter, student), T), ((basketball, team\_sport), T), ((Bob, child), t)\}$$

where t specifies the condition *he drinks milk*. This can explain why in $KP2$ we have both the relation is_a and the relation e_is_a. Really, for $((x, y), t)$ in the relation represented by is_a the corresponding semantics is *x is y IF t*, whereas the meaning for the same pair, in the relation represented by e_is_a, is *every x is y IF t.*

We denote by $R = 2^{Ob \times Ob}$, that is the set of all *binary relations* obtained by means of the objects. Let $f_0 : E \longrightarrow R$ be the mapping

$$f_0(a) = \{(x, y) \in Ob \times Ob \mid \exists t, u, v : ((x, u), (a, t), (y, v)) \in \Gamma\}$$

We remark that $f_0(a)$ is a binary relation and not a conditional binary relation. For example, refering to $KP2$ we obtain:

$$f_0(e\_is\_a) = \{(student, competitor)\}$$
$$f_0(is\_a) = \{(Peter, student), (Alin, student), (George, student), (Mike, student)\}$$

Definition 3. *Let*

$$d = ([(n_1, w_1), \ldots, (n_{k+1}, w_{k+1})], [(a_1, t_1), \ldots, (a_k, t_k)])$$

be a path in G. We say that:

- *the node n_{i_1} is the nearest individual of n_{i_2}, where $1 \leq i_1 \leq i_2 \leq k + 1$, if $w_{i_1} = i$ and does not exist l such that $i_1 < l < i_2$ and (n_l, i) is in d*
- *$t_j[d] = on$, where $1 \leq j \leq k$, if and only if either $t_j = T$ or $t_j(x) = on$ for the nearest individual x of n_j*

We consider a superset E^* of the set E, that is $E \subseteq E^*$, an extension $f : E^* \longrightarrow R$ of the mapping f_0 and a partial binary operation

$$\varphi : E^* \times E^* \longrightarrow E^*$$

that satisfies the following property:

$$f(\varphi(e_1, e_2)) = f(e_1) \circ f(e_2)$$

where $\circ$ is the product operation on R. It is understood that the previous equality holds for every e_1, e_2 such that $\varphi(e_1, e_2)$ is defined.

The pairs (E^*, φ) and $(R, \circ)$ are partial algebras and f becomes a morphism of partial algebras. The existence of these entities can be proven as in the case of the labelled stratified graphs. Details on this problem can be found in [6] and [7].

In order to give an answer to an interrogation we realize a deduction. The answer is a sentence in a natural language. We denote by $\mathcal{S}$ a set of such sentences. To specify the answer we shall use a mapping

$$g : Ob \times E^* \times Ob \times \{on, off\} \longrightarrow \mathcal{S}$$

that specifies the meaning of an element from E^*.
Intuitively, the sentence $g(x, a, y, on)$ will specify the property given by the semantics of $f(a)$, whereas $g(x, a, y, off)$ will specify the contrary property. For example, in the case of $KP2$ we have:

- $g(x, is\_a, y, on) =$ *"x is a y"*
- $g(x, is\_a, y, off) =$ *"x is not a y"*
- $g(x, plays, y, on) =$ *"x plays y"*

that is,

$g(Peter, is\_a, student, on) =$ "Peter is a student"
$g(Peter, is\_a, competitor, off) =$ "Peter is not a competitor".

Definition 4. *A system*

$$(Ob, E^*, \mathcal{C}_{sym}, A, G, \mathcal{S}, \varphi, f, g)$$

is a **conditional scheme** *over G, where $G = (X, \Gamma)$ is a directed graph with conditional relations.*

Definition 5. *Let us consider the path*

$$d = ([(n_1, w_1), \ldots, (n_{k+1}, w_{k+1})], [(a_1, t_1), \ldots, (a_k, t_k)])$$

such that there exists $j \in \{1, \ldots, k+1\}$ such that $w_j = i$. We consider the elements:

$b_1 = a_1$
$\varphi(b_1, a_2) = b_2$
....
$\varphi(b_{k-1}, a_k) = b_k$

We define $ans(d)$ as follows:

- $ans(d) = g(n_1, b_k, n_{k+1}, on)$ *if*

$$t_1[d] = \ldots = t_k[d] = on$$

  and $g(n_1, b_k, n_{k+1}, on)$ is defined
- $ans(d) = g(n_1, b_k, n_{k+1}, off)$ *if there exists t_i such that*

$$t_i[d] = off$$

  and $g(n_1, b_k, n_{k+1}, off)$ is defined
- $ans(d) = unknown$*, otherwise*

Definition 6. *An interrogation for a conditional scheme is a pair $(n_1, n_{k+1}) \in Ob \times Ob$. The answer to the interrogation is the collection of all elements of the set*

$$Ans(n_1, n_{k+1}) = \bigcup_{d \in Path(n_1, n_{k+1})} \{ans(d) \mid ans(d) \neq unknown\}$$

if this is a nonempty set and the answer is **unknown** *otherwise.*

Finally we can give the following **algorithm** to obtain a deduction:

Input: A knowledge piece given in a natural language.
Step 1: Extract the directed graph with conditional relations $G = (X, \Gamma)$
Step 2: Define E^*, φ, f and g
Step 3: Define rules to compute the values **on** or **off** for the elements of $\mathcal{C}_{sym}$
Step 4: Take a pair (n_1, n_{k+1}) of objects; compute $Ans(n_1, n_{k+1})$ and display all the elements of this set if this is a nonempty set; otherwise display **unknown**.

Let us exemplify the computations for $KP2$. We take:

- $Ob = \{Peter, George, Mike, Alin, student, basketball, competitor, team\_sport, mention, prize\}$
- $E = \{friend\_of, brother\_of, is\_a, e\_is\_a, obtains\}$
- $f_0(is\_a) = \{(Peter, student), (George, student), (Mike, student), (Alin, student)\}$
- $f_0(e\_is\_a) = \{(student, competitor)\}$
- $f_0(obtains) = \{(competitor, mention), (competitor, prize)\}$
- $\varphi(is\_a, e\_is\_a) = is\_a$; $\varphi(is\_a, obtains) = obtains$
- $g(x, obtains, y, on)$ = "x obtained a y"; $g(x, obtains, y, off)$ = "x did not obtain a y"
- the rules $R_1, \ldots, R_4$ from Section 3

We can now consider the path d of below:

$$([(Peter, i), (student, a), (competitor, a), (mention, a)],\\ [(is\_a, T), (e\_is\_a, p), (obtains, q)])$$

from $Path(Peter, mention)$. By computation we obtain:

$$\varphi(\varphi(is\_a, e\_is\_a), obtains)) = obtains$$

We have:

- $p[d] = on$ because $p(Peter) = on$ by rule $R_1(Peter)$ and $Peter$ is the nearest individual of $student$
- $q[d] = on$ because $q(Peter) = on$ by rule $R_2(Peter)$ and $Peter$ is the nearest individual of $competitor$

It follows that $ans(d) = g(Peter, obtains, mention, on)$ and an answer to the interrogation $(Peter, mention)$ is *Peter obtained a mention.*

If we restart the computation for the interrogation $(Alin, mention)$ we obtain the answer *Alin did not obtain a mention.*

5 Future work

In a future paper we shall extend the answer function for the case when the path contains only abstract nodes. This will imply an abstract reasoning instead of a particular one based on individual objects. On the other hand we intend to introduce the question *Why ?* after the answer is displayed. For example, if after the last answer of the previous computation we address the question *Why ?*, the system will explain why Alin did not obtain a mention.

References

[1] G. W. Mineau, B. Moulin, J.F. Sowa (Eds), Conceptual Graphs for Knowledge Representation, Lecture Notes in AI 699, Springer-Verlag, Berlin, 1993
[2] M.L. Mugnier, M. Chein (Eds), Conceptual Structures: Theory, Tools, and Applications, Lecture Notes in AI 1453, Springer-Verlag, Berlin, 1998
[3] U. Priss, D. Corbett, G. Angelova (Eds), Conceptual Structures: Integration and Interfaces, 10th Int. Conf. on Conceptual Structures, ICCS 2002, Borovets, Bulgaria, July 15-19, 2002
[4] J.F. Sowa, Conceptual structures- Information Processing in Mind and Machine, Addison-Wesley, 1984
[5] J.F. Sowa, Knowledge-Based Systems, "Special Issue: Conceptual Graphs", 5, (3) (September 1992)
[6] Ţăndăreanu N. (2000), Knowledge Bases with Output, Knowledge and Information Systems, 2, 438-460.
[7] Ţăndăreanu N. (2000), Proving the existence of labelled stratified graphs, Annals of the University of Craiova, Mathematics and Computer Science Series, XXVII, 81-92.
[8] W.M. Tepfenhart, J.P. Dick, J.F. Sowa (Eds), Conceptual Structures: Current Practice, Lecture Notes in AI 835, Springer-Verlag, Berlin, 1994
[9] W. Tepfenhart, W. Cyre (Eds): Conceptual Structures: Standards and Practices, Lecture Notes in AI 1640, Springer-Verlag, Berlin, 1999