# Lattices of labelled ordered trees (II)

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**Abstract**

In this paper the following results are presented: 1) we give a necessary and sufficient condition for the existence of the greatest element in the lattice $Tree_\omega(b_0)/_\approx$ introduced in [2]; 2) we characterize the representatives of the greatest element. All these results will be used in a forthcoming paper to study the properties of the answer function for a knowledge representation and reasoning system based on inheritance property.

**Keywords:** labelled tree, lattice, greatest element

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## 1 Basic notions and results

In this section we summarize the main concepts and results obtained in [2]. We consider a finite set $L$ and a decomposition $L = N_L \cup T_L$, where $N_L \cap T_L = \emptyset$. The elements of $N_L$ are called *nonterminal labels* and those of $T_L$ are called *terminal labels*. The elements of $L$ are called *labels*. A *pairwise mapping* $\omega$ on $L$ is a mapping $\omega : N_L \longrightarrow \bigcup_{k\geq 1}\{k\} \times L^k$. For each $b \in N_L$ we shall denote $\omega(b) = (\omega_1(b), \omega_2(b))$. An $\omega$-labelled tree is a tuple $t = (A, R, h)$, where

- $(A, R)$ is an ordered tree
- $h : A \longrightarrow L$ is a mapping satisfying the following condition: if $[(i, i_1), \ldots, (i, i_s)] \in R$ then $h(i) \in N_L$, $s = \omega_1(h(i))$ and $\omega_2(h(i)) = (h(i_1), \ldots, h(i_s))$

We denote by $1, 2, \ldots, n$ the elements of the set $A$ and these elements are the nodes of $t$. The root of $t$ is denoted by $root(t)$, therefore $root(t) \in A$. In general we suppose $root(t) = 1$.

We consider an element $b_0 \in N_L$ and denote by $Tree_\omega(b_0)$ the set of all $\omega$-labelled trees $t = (A, R, h)$ such that $h(root(t)) = b_0$. If $u = [(i, i_1), \ldots, (i, i_s)] \in R$ then we denote $pr_{r_1,\ldots,r_m}u = [(i, i_{r_1}), \ldots, (i, i_{r_m})]$ where $1 \leq r_1 < r_2 < \ldots < r_m \leq s$. We shall write $v \subseteq u$ if there are $r_1, \ldots, r_m$ such that $v = pr_{r_1,\ldots,r_m}u$.

Let be $t_1 = (A_1, R_1, h_1) \in Tree_\omega(b_0)$ and $t_2 = (A_2, R_2, h_2) \in Tree_\omega(b_0)$. We define the following binary relation on $Tree_\omega(b_0)$: $t_1 \preceq t_2$ if there is an injective mapping $\alpha : A_1 \longrightarrow A_2$ such that:

1) $\alpha(root(t_1)) = root(t_2)$

2) if $u = [(i, i_1), \ldots, (i, i_s)] \in R_1$ then there is $v \in R_2$ such that $\overline{\alpha}(u) \subseteq v$, where

$$\overline{\alpha}(u) = [(\alpha(i), \alpha(i_1)), \ldots, (\alpha(i), \alpha(i_s))]$$

3) $h_1(i) = h_2(\alpha(i))$ for every $i \in A_1$

## 2 The distributive lattice $Tree_\omega(b_0)/\approx$

Let be $t_1, t_2 \in Tree_\omega(b_0)$. We define $t_1 \approx t_2$ iff $t_1 \preceq t_2$ and $t_2 \preceq t_1$. Because $\approx$ is an equivalence relation (see [2]), we can consider the factor set $Tree_\omega(b_0)/_\approx$ of the equivalence classes. For every tree $t \in Tree_\omega(b_0)$ we denote by $[t]$ the equivalence class of $t$.

We define the following relation on $Tree_\omega(b_0)/_\approx$ :

$$[t_1] \ll [t_2] \quad \text{if and only if} \quad t_1 \preceq t_2$$

Let be $t = (A, R, h) \in Tree_\omega(b_0)$. We denote by $S(t)$ the following subset of $\bigcup_{p \geq 1} N^p \times L^p$:

$$((l_1, \ldots, l_s), (b_1, \ldots, b_s)) \in S(t)$$

iff there is $(1, i_1, \ldots, i_s) \in Path(t)$ such that $(1, i_1) \in R^{(l_1)}, \ldots, (i_{s-1}, i_s) \in R^{(l_s)}$ and $h(i_1) = b_1, \ldots, h(i_s) = b_s$.

We have $t_1 \preceq t_2$ if and only if $S(t_1) \subseteq S(t_2)$ and therefore $t_1 \approx t_2$ if and only if $S(t_1) = S(t_2)$ (see [2]).

We define the algebraic operations

$$\vee : Tree_\omega(b_0)/_\approx \times Tree_\omega(b_0)/_\approx \longrightarrow Tree_\omega(b_0)/_\approx$$

$$\wedge : Tree_\omega(b_0)/_\approx \times Tree_\omega(b_0)/_\approx \longrightarrow Tree_\omega(b_0)/_\approx$$

as follows:

$[t_1] \vee [t_2] = [t]$, where $S(t) = S(t_1) \cup S(t_2)$

$[t_1] \wedge [t_2] = [t]$, where $S(t) = S(t_1) \cap S(t_2)$

In [2] is shown that $(Tree_\omega(b_0)/_\approx, \vee, \wedge)$ is a lattice. Moreover, using the corresponding properties from the set theory we deduce now that the lattice $Tree_\omega(b_0)/_\approx$ is a distributive one, a property which is proved in the next proposition.

**Proposition 2.1** *The lattice $(Tree_\omega(b_0)/_\approx, \vee, \wedge)$ is distributive.*

**Proof.** Really, if we denote $\alpha = [t_1] \wedge ([t_2] \vee [t_3])$, $\beta = ([t_1] \wedge [t_2]) \vee ([t_1] \wedge [t_3])$ and we consider $p \in Tree_\omega(b_0)$ then the following sentences are equivalent:

- $p \in \alpha$
- $S(p) = S(t_1) \cap (S(t_2) \cup S(t_3)) = (S(t_1) \cap S(t_2)) \cup (S(t_1) \cap S(t_3))$
- $p \in \beta$

It follows that $\alpha = \beta$. ■

## 3 The greatest element of the lattice $Tree_\omega(b_0)/_\approx$

In this section we give a necessary and sufficient condition for the existence of the greatest element in the lattice $Tree_\omega(b_0)/\approx$. Moreover, we give a characterization for the greatest element.

Let be $L = N_L \cup T_L$ a set of labels, $\omega : N_L \longrightarrow \bigcup_{k\geq 1}\{k\} \times L^k$ a pairwise mapping and $b_0 \in N_L$. Let us consider $X \subseteq L$ and define:

$$U(X) = \{y \in L \mid \exists a \in X \cap N_L, \exists i \in \{1, \ldots, \omega_1(a)\} : y = pr_i\omega_2(a)\}$$

where $pr_i\omega_2(a)$ denotes the $i^{th}$ component of the element $\omega_2(a)$.

For an arbitrary element $b \in L$ we define the sequence:

$$\begin{cases} S_b^0 = U(\{b\}) \\ \\ S_b^{n+1} = S_b^n \cup U(S_b^n), \qquad n \geq 0 \end{cases}$$

We observe that for $b \in T_L$ we obtain $S_b^0 = S_b^1 = \ldots = \emptyset$. For $b \in N_L$, the sequence $\{S_b^n\}_n$ is an increasing one:

$$\emptyset \neq S_b^0 \subseteq S_b^1 \subseteq \ldots \subseteq S_b^n \subseteq \ldots \subseteq L$$

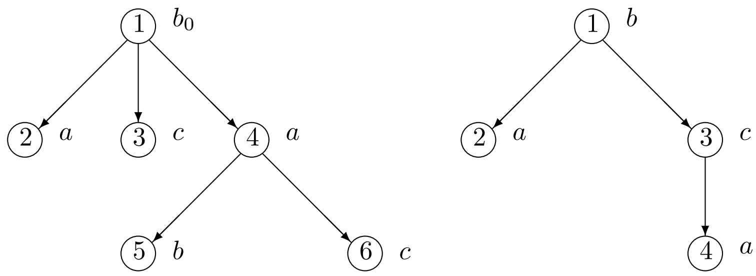
If $S_b^0 \subseteq T_L$ then $S_b^0 = S_b^1 = \ldots$ and we take $m(b) = 0$. Otherwise, since $L$ is a finite set there is $m(b) \geq 1$ such that $S_b^0 \subset \ldots \subset S_b^{m(b)} = S_b^{m(b)+1} = \ldots$. In other words, $m(b) = 0$ for $b \in T_L$ and $m(b) \geq 1$ for $b \in N_L$. This notation permits us to define $T : L \longrightarrow 2^L$ as follows:

$$T(b) = \begin{cases} \emptyset & if \quad b \in T_L \\ \\ S_b^{m(b)} & if \quad b \in N_L \end{cases}$$

The mapping $T$ is used to characterize the existence of the greatest element in $Tree_\omega(b_0)/_\approx$. We shall obtain first several auxiliary properties.

We consider $b_0, b \in L$. Let be $t_1 = (A_1, R_1, h_1) \in Tree_\omega(b_0)$ and $t_2 = (A_2, R_2, h_2)$ $\in Tree_\omega(b)$ such that for some leaf $i$ of $t_1$ we have $h(i) = b$. We suppose $A_1 = \{1, \ldots, n_1\}$ and $A_2 = \{1, \ldots, n_2\}$. We denote by $t_1 \oplus_i t_2 = (A, R, h)$ the following tree:

- $A = A_1 \cup \{n_1 + 1, \ldots, n_1 + n_2 - 1\}$
- $R = R_1 \cup R_2'$ where $R_2'$ is defined as follows:
  - $[(i, m_1 + n_1 - 1), \ldots, (i, m_p + n_1 - 1)] \in R_2'$ iff $[(1, m_1), \ldots, (1, m_p)] \in R_2$
  - $[(n_1 + j - 1, n_1 + j_1 - 1), \ldots, (n_1 + j - 1, n_1 + j_s - 1)] \in R_2'$ if and only if $j \neq 1$ and $[(j, j_1), \ldots, (j, j_s)] \in R_2$
- $h(k) = \begin{cases} h_1(k) & if \quad k \in \{1, \ldots, n_1\} \\ \\ h_2(k - n_1 + 1) & if \quad k \in \{n_1 + 1, \ldots, n_1 + n_2 - 1\} \end{cases}$




tree $t_1$ tree $t_2$

Figure 1: Two labelled trees

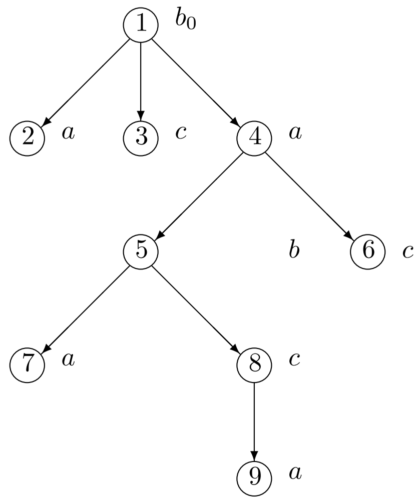



Figure 2: The tree $t_1 \oplus_5 t_2$

This construction is illustrated in **Figure 1** and **Figure 2**. We consider $A_1 = \{1, 2, \ldots, 6\}$ and $A_2 = \{1, 2, 3, 4\}$ and the trees $t_1$ and $t_2$ specified in **Figure 1**. Taking $i = 5$ and applying this construction we obtain the tree $t_1 \oplus_5 t_2 \in Tree_\omega(b_0)$ represented in **Figure 2**.

The following properties can be verified immediately:

- $t_1 \oplus_i t_2 \in Tree_\omega(b_0)$ if $t_1 \in Tree_\omega(b_0)$
- $t_1 \preceq t_1 \oplus_i t_2$ for every $t_2$ and every $i$
- every $j \in \{1, \ldots, n_1\} \setminus \{i\}$, which is a leaf in $t_1$, is also a leaf in $t_1 \oplus_i t_2$
- if $j$ is a leaf of $t_2$ then $n_1 + j - 1$ is a leaf of $t_1 \oplus_i t_2$

The next proposition gives a characterization for the elements of the set $T(b_0)$, where $b_0 \in N_L$. The proof uses the above construction for $t_1 \oplus_i t_2$.

**Proposition 3.1** *Let be $b_0 \in N_L$. We have $b \in T(b_0)$ iff there is $t = (A, R, h) \in Tree_\omega(b_0)$ such that $h(i) = b$ for some leaf $i$ of $t$.*

**Proof.** By induction on $n$ we shall prove that if $b \in S^n_{b_0}$ then there is $t \in Tree_\omega(b_0)$ such that $h(i) = b$ for some leaf $i$ of $t$. Obviously this assertion is true for $n = 0$. Suppose the assertion is true for $n = m$. Let us consider $b \in S^{m+1}_{b_0} \setminus S^m_{b_0} = U(S^m_{b_0})$. There is $a \in S^m_{b_0} \cap N_L$ such that $\omega_2(a) = (b_1, \ldots, b_{\omega_1(a)})$ and $b = b_i$ for some $i \in \{1, \ldots, \omega_1(a)\}$. By inductive assumption there is $t_a = (A, R, h) \in Tree_\omega(b_0)$ such that $h(j) = a$ for some leaf $j$ of $t_a$. We consider the tree $t_1 = (A_1, R_1, h_1)$, where $A_1 = \{1, \ldots, \omega_1(a) + 1\}$, $R_1 = \{[(1, 2), \ldots, (1, \omega_1(a) + 1)]\}$ and $h_1(1) = a$, $h_1(2) = b_1$, $\ldots$, $h_1(\omega_1(a) + 1) = b_{\omega_1(a)}$. The tree $t_a \oplus_j t_1$ satisfies the conditions: $t_a \oplus_j t_1 \in Tree_\omega(b_0)$ and there is a leaf which is labelled by $b$.
In order to prove the converse property, we consider $t = (A, R, h) \in Tree_\omega(b_0)$ and we prove by induction on $k$ that if $i \in level_k(t)$ then $h(i) \in T(b_0)$. Obviously if $i \in level_1(t)$ then $h(i) \in S^0_{b_0}$. Suppose the property is true for $k$ and let be $i \in level_{k+1}(t)$. There is a path $(1, i_1, \ldots, i_k, i)$ in $t$. By inductive assumption $h(i_k) \in T(b_0)$. There is $m \geq 0$ such that $h(i_k) \in S^m_{b_0}$. Let $[(i_k, j_1), \ldots, (i_k, j_s)] \in R$ such that $i = j_r$ for some $r \in \{1, \ldots, s\}$. We have $s = \omega_1(h(i_k))$ and $\omega_2(h(i_k)) = (h(j_1), \ldots, h(j_s))$. Thus $h(j_r) \in S^{m+1}_{b_0} \subseteq T(b_0)$, that is $h(i) \in T(b_0)$. ∎

**Proposition 3.2** *If $b \in T(c)$ and $c \in T(a)$ then $b \in T(a)$.*

**Proof.** By **Proposition 3.1** it follows that there are $t_1 \in Tree_\omega(c)$ and $t_2 \in Tree_\omega(a)$ such that some leaf $i_1$ of $t_1$ is labelled by $b$ and some leaf $i_2$ of $t_2$ is labelled by $c$. We take the tree $t_2 \oplus_{i_2} t_1 \in Tree_\omega(a)$ and by **Proposition 3.1** it follows that $b \in T(a)$. ∎

**Proposition 3.3** *If the lattice $(Tree_\omega(b_0)/_\approx, \vee, \wedge)$ has a greatest element then $b \notin T(b)$ for every $b \in \{b_0\} \cup T(b_0)$.*

**Proof.**

Suppose the lattice has a greatest element. By contrary we assume that $b \in T(b)$ for some $b \in \{b_0\} \cup T(b_0)$. Two cases will be analyzed:

1) Suppose $b \in T(b_0)$
If $b \in T_L$ then $T(b) = \emptyset$ and thus we have $b \notin T(b)$. We suppose now $b \in N_L$. From $b \in T(b_0) \cap T(b)$ and by **Proposition 3.1** it follows that there are $t_0 = (A_0, R_0, h_0) \in Tree_\omega(b_0)$ and $t_1 = (A_1, R_1, h_1) \in Tree_\omega(b)$ such that $h_0(i_0) = h_1(i) = b$ for some leaves $i_0$ and $i$ in $t_0$, respectively $t_1$. We consider the tree $t_0 \oplus_{i_0} t_1 \in Tree_\omega(b_0)$. There is a leaf $i_1$ in this tree having the label $b$. We take the tree $(t_0 \oplus_{i_0} t_1) \oplus_{i_1} t_1$ and we repeat this step. Thus we obtain a sequence of trees:

$$\begin{cases} \alpha_0 = t_0 \oplus_{i_0} t_1 \\ \\ \alpha_{j+1} = \alpha_j \oplus_{i_{j+1}} t_1, \qquad j \geq 0 \end{cases}$$

such that each $\alpha_j$ contains a leaf labelled by $b$. If $B_0, B_1, \ldots$ are the corresponding sets of the nodes for these trees then $Card(B_0) < Card(B_1) < \ldots$. Let be $[t_g]$ the greatest element of $Tree_\omega(b_0)/_\approx$. If $A_g$ is the node set of $t_g$ then we have $Card(B_0) < Card(B_1) < \ldots < Card(A_g)$, which is not true since $A_g$ is a finite set.

2) $b = b_0$
We take $t_1 = t_0$ and we proceed as above.

Thus, the assumption $b \in T(b)$ for some $b \in \{b_0\} \cup T(b_0)$ is not true. ∎

**Proposition 3.4** *Let be $b_0 \in N_L$. Suppose $b \notin T(b)$ for every $b \in \{b_0\} \cup T(b_0)$. We define recursively the following sets $Q_1, Q_2, \ldots$ as follows:*

- $((l), (b)) \in Q_1$ *iff* $l \in \{1, \ldots, \omega_1(b_0)\}$ *and* $b = pr_l\omega_2(b_0)$
- $((l_1, \ldots, l_k, l), (b_1, \ldots, b_k, b)) \in Q_{k+1}$ *if and only if* $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Q_k$, $l \in \{1, \ldots, \omega_1(b_k)\}$, $b = pr_l\omega_2(b_k)$

*Then the following properties are satisfied:*

1) $Q_{n(l)+1} = Q_{n(l)+2} = \ldots = \emptyset$, *where* $n_l = Card(N_L)$

2) $X = \bigcup_{k=1}^{n(l)} Q_k$ *satisfies the* $\omega - b_0$*-conditions*

3) *For every $Y$ satisfying the $\omega - b_0$-conditions we have $Y \subseteq X$*

**Proof.** For every $k \geq 1$, if $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Q_k$ then $b_j \in T(b_i)$ for every $i, j$ such that $0 \leq i < j \leq k$. We prove this assertion by induction on $k$. For $k = 1$, if $((l), (b)) \in Q_1$ then $b = pr_l\omega_2(b_0)$ and $l \in \{1, \ldots, \omega_1(b_0)\}$. Then $b \in U(\{b_0\}) \subseteq T(b_0)$. Assuming the assertion is true for $k$, if $((l_1, \ldots, l_k, l), (b_1, \ldots, b_k, b)) \in Q_{k+1}$ then $b_j \in T(b_i)$ for $0 \leq i < j \leq k$ and $b = pr_l\omega_2(b_k)$, where $l \in \{1, \ldots, \omega_1(b_k)\}$. Then $b \in U(\{b_k\}) \subseteq T(b_k)$. By inductive assumption $b_k \in T(b_i)$ for every $i \in \{0, \ldots, k-1\}$. By **Proposition 3.2** it follows that $b \in T(b_i)$ for each $i \in \{0, \ldots, k-1\}$. Thus, if $((l_1, \ldots, l_k, l_{k+1}), (b_1, \ldots, b_k, b_{k+1})) \in Q_{k+1}$ then $T(b_i) \neq \emptyset$ for every $i \in \{1, \ldots, k\}$, therefore $b_1, \ldots, b_k \in N_L$. The sequence $b_0, b_1, \ldots, b_k$ satisfies the condition $b_i \neq b_j$ for $i \neq j$. Really, if $b_i = b_j = b$ for some $i < j$ then we have $b \in T(b)$, where $b \in \{b_0\} \cup T(b_0)$, which is not true. Therefore, if $Q_{k+1} \neq \emptyset$ then $k + 1 \leq n_l$. If we

suppose that $Q_{n(l)+1} \neq \emptyset$ then there is $((k_1, \ldots, k_{n(l)+1}), (b_1, \ldots, b_{n(l)+1})) \in Q_{n(l)+1}$. This implies that $b_0, b_1, \ldots, b_{n(l)}$ are distinct elements of $N_L$, which is not true. Thus the first property is proved.
Obviously $X = \bigcup_{k=1}^{n(l)} Q_k$ satisfies the $\omega - b_0$-conditions. Let $Y$ be a set satisfying the $\omega - b_0$-conditions. We verify by induction $k$ that for every $k \geq 1$ , if $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Y$ then $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Q_k$. For $k = 1$ the assertion is true. We assume the assertion is true for $k$ and let be $((l_1, \ldots, l_k, l), (b_1, \ldots, b_k, b)) \in Y$. Since $Y$ satisfies the $\omega - b_0$-conditions we have $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Y$, $l \in \{1, \ldots, \omega_1(b_k)\}$, $b = pr_l\omega_2(b_k)$ and $((l_1, \ldots, l_k, m), (b_1, \ldots, b_k, pr_m\omega_2(b_k))) \in Y$ for $m \in \{1, \ldots, \omega_1(b_k)\}$. By inductive assumption we have $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Q_k$ and by the definition of $Q_{k+1}$ we have $((l_1, \ldots, l_k, m), (b_1, \ldots, b_k, pr_m\omega_2(b_k))) \in Q_{k+1}$ for every $m \in \{1, \ldots, \omega_1(b_k)\}$. Particularly we have $((l_1, \ldots, l_k, l), (b_1, \ldots, b_k, b)) \in Q_{k+1}$. Thus $Y \subseteq \bigcup_{k \geq 1} Q_k = \bigcup_{k=1}^{n(l)} Q_k = X$. ■

**Proposition 3.5** *Suppose that* $b_0 \in N_L$ *and* $b \notin T(b)$ *for every* $b \in \{b_0\} \cup T(b_0)$. *Then the lattice* $(Tree_\omega(b_0)/_\approx, \vee, \wedge)$ *contains a greatest element.*

**Proof.** Let us consider $X = \bigcup_{k=1}^{n(l)} Q_k$ from **Proposition 3.4**. Since $X$ satisfies the $\omega - b_0$-conditions it follows that there is $t_g \in Tree_\omega(b_0)$ such that $S(t_g) = X$. Let be now $[t] \in Tree_\omega(b_0)/_\approx$. $S(t)$ satisfies the $\omega - b_0$-conditions, therefore $S(t) \subseteq S(t_g)$. This implies $t \leq t_g$, therefore $[t] \preceq [t_g]$. ■

**Proposition 3.6** *Suppose* $(Tree_\omega(b_0)/\approx, \vee, \wedge)$ *contains a greatest element. If* $t_0 = (A_0, R_0, h_0) \in Tree_\omega(b_0)$ *is such that for every leaf* $i$ *of* $t_0$ *we have* $h_0(i) \in T_L$ *then for every* $t \in Tree_\omega(b_0)$ *we have* $t \preceq t_0$.

**Proof.** Suppose that for every leaf $i$ of $t_0$ we have $h_0(i) \in T_L$. Let be $t = (A, R, h) \in Tree_\omega(b_0)$. We shall verify that $S(t) \subseteq S(t_0)$, which will show that $t \preceq t_0$. Let be $((l_1, \ldots, l_s), (b_1, \ldots, b_s)) \in S(t)$. There is $(1, i_1, \ldots, i_s) \in Path(t)$ such that $(1, i_1) \in R^{(l_1)}$, $(i_1, i_2) \in R^{(l_2)}$, ..., $(i_{s-1}, i_s) \in R^{(l_s)}$, $h(i_1) = b_1$, ..., $h(i_s) = b_s$. Only the nodes labelled by nonterminal labels may have direct descendants, therefore $b_1, \ldots, b_{s-1} \in N_L$. In order to simplify the notation we denote $\omega_1(b_j) = n_j$ for $j \in \{0, \ldots, s-1\}$. For every $j \in \{1, \ldots, s\}$ there is $u_j = [(i_{j-1}, k_1^{(j)}), \ldots, (i_{j-1}, k_{n_{j-1}}^{(j)})] \in R$ such that $i_j = k_{l_j}^{(j)}$, where $i_0 = 1$. Since $t_0$ is an $\omega$-labelled tree it folows that there is $v_1 = [(1, r_1^{(1)}), \ldots, (1, r_{n_0}^{(1)})] \in R_0$ such that $h_0(r_{l_1}^{(1)}) = b_1$. We denote $j_1 = r_{l_1}^{(1)}$. By induction on $p$ we prove that for every $p \in \{1, \ldots, s\}$ there is $v_p = [(j_{p-1}, r_1^{(p)}), \ldots, (j_{p-1}, r_{n_{p-1}}^{(p)})] \in R_0$ such that $h_0(r_{l_p}^{(p)}) = b_p$, where $j_p = r_{l_p}^{(p)}$ and $j_0 = 1$. For $p = 1$ the property is verified. We assume the property is verified for some $p \in \{1, \ldots, s-1\}$. Since $b_p \in N_L$ and $h_0(j_p) = b_p$ it follows that $j_p$ has $n_p$ direct descendants in $t_0$. Thus there is $v_{p+1} = [(j_p, r_1^{(p+1)}), \ldots, (j_p, r_{n_p}^{(p+1)})] \in R_0$ such that $h_0(r_{l_{p+1}}^{(p+1)}) = b_{p+1}$. We take $j_{p+1} = r_{l_{p+1}}^{(p+1)}$. In this way we obtain $(1, j_1, \ldots, j_s) \in Path(t_0)$ such that $(1, j_1) \in R_0^{(l_1)}$, $(j_1, j_2) \in R_0^{(l_2)}$, ..., $(j_{s-1}, j_s) \in R_0^{(l_s)}$, $h_0(j_1) = b_1$, ..., $h_0(j_s) = b_s$. Therefore $((l_1, \ldots, l_s), (b_1, \ldots, b_s)) \in S(t_0)$. ■

**Corollary 3.1** *Suppose* $(Tree_\omega(b_0)/\approx, \vee, \wedge)$ *contains a greatest element. If* $t_1 = (A_1, R_1, h_1) \in Tree_\omega(b_0)$ *and* $t_2 = (A_2, R_2, h_2) \in Tree_\omega(b_0)$ *are such that* $h_1(i) \in T_L$ *and* $h_2(j) \in T_L$ *for each leaves* $i$ *and* $j$ *then* $t_1 \approx t_2$, *therefore* $[t_1] = [t_2]$.

**Proof.** Really, by **Proposition 3.6** we have $t_1 \preceq t_2$ and $t_2 \preceq t_1$, therefore $t_1 \approx t_2$. ∎

**Remark 3.1** *Suppose* $(Tree_\omega(b_0)/\approx, \vee, \wedge)$ *contains a greatest element. The tree* $t_0 = (A_0, R_0, h_0) \in Tree_\omega(b_0)$ *is a representative of the greatest element if and only if for every leaf* $i$ *of* $t_0$ *we have* $h_0(i) \in T_L$.

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