Lattices of labelled ordered trees (II)

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Abstract

In this paper the following results are presented: 1) we give a necessary and sufficient condition for the existence of the greatest element in the lattice \( \text{Tree}_\omega(b_0) / \approx \) introduced in [2]; 2) we characterize the representatives of the greatest element. All these results will be used in a forthcoming paper to study the properties of the answer function for a knowledge representation and reasoning system based on inheritance property.

Keywords: labelled tree, lattice, greatest element

AMS Subject Classification: 06B05, 06A12

1 Basic notions and results

In this section we summarize the main concepts and results obtained in [2]. We consider a finite set \( L \) and a decomposition \( L = N_L \cup T_L \), where \( N_L \cap T_L = \emptyset \). The elements of \( N_L \) are called \textit{nonterminal labels} and those of \( T_L \) are called \textit{terminal labels}. The elements of \( L \) are called \textit{labels}. A \textit{pairwise mapping} \( \omega \) on \( L \) is a mapping \( \omega : N_L \rightarrow \bigcup_{k \geq 1} [k] \times L^k \). For each \( b \in N_L \) we shall denote \( \omega(b) = (\omega_1(b), \omega_2(b)) \).

An \( \omega \)-labelled tree is a tuple \( t = (A, R, h) \), where

- \( (A, R) \) is an ordered tree
- \( h : A \rightarrow L \) is a mapping satisfying the following condition: if \( [(i, i_1), \ldots, (i, i_n)] \in R \) then \( h(i) \in N_L, s = \omega_1(h(i)) \) and \( \omega_2(h(i)) = (h(i_1), \ldots, h(i_n)) \)

We denote by 1, 2, \ldots, \( n \) the elements of the set \( A \) and these elements are the nodes of \( t \). The root of \( t \) is denoted by \( \text{root}(t) \), therefore \( \text{root}(t) \in A \). In general we suppose \( \text{root}(t) = 1 \).

We consider an element \( b_0 \in N_L \) and denote by \( \text{Tree}_\omega(b_0) \) the set of all \( \omega \)-labelled trees \( t = (A, R, h) \) such that \( h(\text{root}(t)) = b_0 \). If \( u = [(i, i_1), \ldots, (i, i_s)] \in R \) then we denote \( pr_{r_1}, \ldots, r_m u = [(i, r_1), \ldots, (i, r_m)] \) where \( 1 \leq r_1 < r_2 < \ldots < r_m \leq s \). We shall write \( v \subseteq u \) if there are \( r_1, \ldots, r_m \) such that \( v = pr_{r_1}, \ldots, r_m u \).

Let be \( t_1 = (A_1, R_1, h_1) \in \text{Tree}_\omega(b_0) \) and \( t_2 = (A_2, R_2, h_2) \in \text{Tree}_\omega(b_0) \). We define the following binary relation on \( \text{Tree}_\omega(b_0) \): \( t_1 \preceq t_2 \) if there is an injective mapping \( \alpha : A_1 \rightarrow A_2 \) such that:

1) \( \alpha(\text{root}(t_1)) = \text{root}(t_2) \)

2) if \( u = [(i, i_1), \ldots, (i, i_s)] \in R_1 \) then there is \( v \in R_2 \) such that \( \pi(u) \subseteq v \), where

\[
\pi(u) = [(\alpha(i), \alpha(i_1)), \ldots, (\alpha(i), \alpha(i_s))] 
\]
3) \( h_1(i) = h_2(\alpha(i)) \) for every \( i \in A_1 \)

### 2 The distributive lattice \( \text{Tree}_\omega(b_0)/\approx \)

Let be \( t_1, t_2 \in \text{Tree}_\omega(b_0) \). We define \( t_1 \approx t_2 \) iff \( t_1 \preceq t_2 \) and \( t_2 \preceq t_1 \). Because \( \approx \) is an equivalence relation (see [2]), we can consider the factor set \( \text{Tree}_\omega(b_0)/\approx \) of the equivalence classes. For every tree \( t \in \text{Tree}_\omega(b_0) \) we denote by \( [t] \) the equivalence class of \( t \).

We define the following relation on \( \text{Tree}_\omega(b_0)/\approx \):

\[
[t_1] \ll [t_2] \quad \text{if and only if} \quad t_1 \preceq t_2
\]

Let be \( t = (A, R, h) \in \text{Tree}_\omega(b_0) \). We denote by \( S(t) \) the following subset of \( \bigcup_{p \geq 1} N^p \times L^p \):

\[
((l_1, \ldots, l_s), (b_1, \ldots, b_s)) \in S(t) \quad \text{iff} \quad \text{there is} \quad (1, i_1, \ldots, i_s) \in \text{Path}(t) \quad \text{such that} \quad (1, i_1) \in R^{l_1}, \ldots, (i_{s-1}, i_s) \in R^{l_s} \quad \text{and} \quad h(i_1) = b_1, \ldots, h(i_s) = b_s.
\]

We have \( t_1 \preceq t_2 \) if and only if \( S(t_1) \subseteq S(t_2) \) and therefore \( t_1 \approx t_2 \) if and only if \( S(t_1) = S(t_2) \) (see [2]).

We define the algebraic operations

\[
\lor : \text{Tree}_\omega(b_0)/\approx \times \text{Tree}_\omega(b_0)/\approx \longrightarrow \text{Tree}_\omega(b_0)/\approx
\]

\[
\land : \text{Tree}_\omega(b_0)/\approx \times \text{Tree}_\omega(b_0)/\approx \longrightarrow \text{Tree}_\omega(b_0)/\approx
\]

as follows:

\[
[t_1] \lor [t_2] = [t], \quad \text{where} \quad S(t) = S(t_1) \cup S(t_2)
\]

\[
[t_1] \land [t_2] = [t], \quad \text{where} \quad S(t) = S(t_1) \cap S(t_2)
\]

In [2] is shown that \( (\text{Tree}_\omega(b_0)/\approx, \lor, \land) \) is a lattice. Moreover, using the corresponding properties from the set theory we deduce now that the lattice \( \text{Tree}_\omega(b_0)/\approx \) is a distributive one, a property which is proved in the next proposition.

**Proposition 2.1** The lattice \( (\text{Tree}_\omega(b_0)/\approx, \lor, \land) \) is distributive.

**Proof.** Really, if we denote \( \alpha = [t_1] \land ([t_2] \lor [t_3]) \), \( \beta = ([t_1] \land [t_2]) \lor ([t_1] \land [t_3]) \) and we consider \( p \in \text{Tree}_\omega(b_0) \) then the following sentences are equivalent:

- \( p \in \alpha \)
- \( S(p) = S(t_1) \cap (S(t_2) \cup S(t_3)) = (S(t_1) \cap S(t_2)) \cup (S(t_1) \cap S(t_3)) \)
- \( p \in \beta \)

It follows that \( \alpha = \beta \). □
3 The greatest element of the lattice $\text{Tree}_\omega(b_0)/\approx$

In this section we give a necessary and sufficient condition for the existence of the greatest element in the lattice $\text{Tree}_\omega(b_0)/\approx$. Moreover, we give a characterization for the greatest element.

Let be $L = N_L \cup T_L$ a set of labels, $\omega : N_L \rightarrow \bigcup_{k \geq 1} [k] \times L^k$ a pairwise mapping and $b_0 \in N_L$. Let us consider $X \subseteq L$ and define:

$$U(X) = \{ y \in L \mid \exists a \in X \cap N_L, \exists i \in \{1, \ldots, \omega_1(a)\} : y = pr_i(\omega_2(a))\}$$

where $pr_i(\omega_2(a))$ denotes the $i^{th}$ component of the element $\omega_2(a)$.

For an arbitrary element $b \in L$ we define the sequence:

$$S^0_b = U(\{ b \})$$

$$S^{n+1}_b = S^n_b \cup U(S^n_b), \quad n \geq 0$$

We observe that for $b \in T_L$ we obtain $S^0_b = S^1_b = \ldots = \emptyset$. For $b \in N_L$, the sequence $\{S^n_b\}_n$ is an increasing one:

$$\emptyset \neq S^0_b \subseteq S^1_b \subseteq \ldots \subseteq S^n_b \subseteq \ldots \subseteq L$$

If $S^0_b \subseteq T_L$ then $S^0_b = S^1_b = \ldots$ and we take $m(b) = 0$. Otherwise, since $L$ is a finite set there is $m(b) \geq 1$ such that $S^0_b \subseteq \ldots \subseteq S^{m(b)}_b = S^{m(b)+1}_b = \ldots$. In other words, $m(b) = 0$ for $b \in T_L$ and $m(b) \geq 1$ for $b \in N_L$. This notation permits us to define $T : L \rightarrow 2^L$ as follows:

$$T(b) = \begin{cases} \emptyset & \text{if } b \in T_L \\ S^{m(b)}_b & \text{if } b \in N_L \end{cases}$$

The mapping $T$ is used to characterize the existence of the greatest element in $\text{Tree}_\omega(b_0)/\approx$. We shall obtain first several auxiliary properties.

We consider $b_0, b \in L$. Let be $t_1 = (A_1, R_1, h_1) \in \text{Tree}_\omega(b_0)$ and $t_2 = (A_2, R_2, h_2) \in \text{Tree}_\omega(b)$ such that for some leaf $i$ of $t_1$ we have $h(i) = b$. We suppose $A_1 = \{1, \ldots, n_1\}$ and $A_2 = \{1, \ldots, n_2\}$. We denote by $t_1 \oplus_i t_2 = (A, R, h)$ the following tree:

- $A = A_1 \cup \{ n_1 + 1, \ldots, n_1 + n_2 - 1 \}$
- $R = R_1 \cup R'_2$ where $R'_2$ is defined as follows:
  - $[(i, m_1 + n_1 - 1), \ldots, (i, m_p + n_1 - 1)] \in R'_2$ iff $[(1, m_1), \ldots, (1, m_p)] \in R_2$
  - $[(n_1 + j - 1, n_1 + j_1 - 1), \ldots, (n_1 + j - 1, n_1 + j_k - 1)] \in R'_2$ if and only if $j \neq 1$ and $[(j, j_1), \ldots, (j, j_k)] \in R_2$
- $h(k) = \begin{cases} h_1(k) & \text{if } k \in \{1, \ldots, n_1\} \\ h_2(k - n_1 + 1) & \text{if } k \in \{n_1 + 1, \ldots, n_1 + n_2 - 1\} \end{cases}$
Figure 1: Two labelled trees

Figure 2: The tree $t_1 \oplus_5 t_2$
The following properties can be verified immediately:

- \( t_1 \oplus_i t_2 \in \text{Tree}_\omega(b_0) \) if \( t_1 \in \text{Tree}_\omega(b_0) \)
- \( t_1 \preceq t_1 \oplus_i t_2 \) for every \( t_2 \) and every \( i \)
- every \( j \in \{1, \ldots, n_1\} \setminus \{i\} \), which is a leaf in \( t_1 \), is also a leaf in \( t_1 \oplus_i t_2 \)
- if \( j \) is a leaf of \( t_2 \) then \( n_1 + j - 1 \) is a leaf of \( t_1 \oplus_i t_2 \)

The next proposition gives a characterization for the elements of the set \( T(b_0) \), where \( b_0 \in N_L \). The proof uses the above construction for \( t_1 \oplus_i t_2 \).

**Proposition 3.1** Let be \( b_0 \in N_L \). We have \( b \in T(b_0) \) iff there is \( t = (A, R, h) \in \text{Tree}_\omega(b_0) \) such that \( h(i) = b \) for some leaf \( i \) of \( t \).

**Proof.** By induction on \( n \) we shall prove that if \( b \in S_{b_0}^n \) then there is \( t \in \text{Tree}_\omega(b_0) \) such that \( h(i) = b \) for some leaf \( i \) of \( t \). Obviously this assertion is true for \( n = 0 \). Suppose the assertion is true for \( n = m \). Let us consider \( b \in S_{b_0}^{m+1} \setminus S_{b_0}^m = U(S_{b_0}^m) \). There is \( a \in S_{b_0}^m \cap N_L \) such that \( \omega_1(a) = (b_1, \ldots, b_{\omega_1(a)}) \) and \( b = b_i \) for some \( i \in \{1, \ldots, \omega_1(a)\} \). By inductive assumption there is \( t_a = (A, R, h) \in \text{Tree}_\omega(b_0) \) such that \( h(j) = a \) for some leaf \( j \) of \( t_a \). We consider the tree \( t_1 = (A_1, R_1, h_1) \), where \( A_1 = \{1, \ldots, \omega_1(a) + 1\}, R_1 = \{\{(1,2), \ldots, (1, \omega_1(a) + 1)\}\} \) and \( h_1(1) = a, h_1(2) = b_1, \ldots, h_1(\omega_1(a) + 1) = b_{\omega_1(a)} \). The tree \( t_a \oplus_j t_1 \) satisfies the conditions: \( t_a \oplus_j t_1 \in \text{Tree}_\omega(b_0) \) and there is a leaf which is labelled by \( b \).

In order to prove the converse property, we consider \( t = (A, R, h) \in \text{Tree}_\omega(b_0) \) and we prove by induction on \( k \) that if \( i \in \text{level}_k(t) \) then \( h(i) \in T(b_0) \). Obviously if \( i \in \text{level}_1(t) \) then \( h(i) \in S_{b_0}^0 \). Suppose the property is true for \( k \) and let be \( i \in \text{level}_{k+1}(t) \). There is a path \((1, i_1, \ldots, i_k, i)\) in \( t \). By inductive assumption \( h(i_k) \in T(b_0) \). There is \( m \geq 0 \) such that \( h(i_k) \in S_{b_0}^m \). Let \( [(i_k, j_1), \ldots, (i_k, j_s)] \in R \) such that \( i = j_r \) for some \( r \in \{1, \ldots, s\} \). We have \( s = \omega_1(h(i_k)) \) and \( \omega_2(h(i_k)) = (h(j_1), \ldots, h(j_s)) \). Thus \( h(j_r) \in S_{b_0}^{m+1} \subseteq T(b_0) \), that is \( h(i) \in T(b_0) \).

**Proposition 3.2** If \( b \in T(c) \) and \( c \in T(a) \) then \( b \in T(a) \).

**Proof.** By Proposition 3.1 it follows that there are \( t_1 \in \text{Tree}_\omega(c) \) and \( t_2 \in \text{Tree}_\omega(a) \) such that some leaf \( i_1 \) of \( t_1 \) is labelled by \( b \) and some leaf \( i_2 \) of \( t_2 \) is labelled by \( c \). We take the tree \( t_2 \oplus_{i_2} t_1 \in \text{Tree}_\omega(a) \) and by Proposition 3.1 it follows that \( b \in T(a) \).

**Proposition 3.3** If the lattice \( \text{Tree}_\omega(b_0)/\sim, \lor, \land \) has a greatest element then \( b \not\in T(b) \) for every \( b \in \{b_0\} \cup T(b_0) \).

**Proof.** Suppose the lattice has a greatest element. By contrary we assume that \( b \in T(b) \) for some \( b \in \{b_0\} \cup T(b_0) \). Two cases will be analyzed:
1) Suppose $b \in T(b_0)$
If $b \in T_L$ then $T(b) = \emptyset$ and thus we have $b \not\in T(b)$. We suppose now $b \in N_L$. From $b \in T(b_0) \cap T(b)$ and by Proposition 3.1 it follows that there are $t_0 = (A_0, R_0, h_0) \in Tree_{\omega}(b_0)$ and $t_1 = (A_1, R_1, h_1) \in Tree_{\omega}(b)$ such that $h_0(i_0) = h_1(i) = b$ for some leaves $i_0$ and $i$ in $t_0$, respectively $t_1$. We consider the tree $t_0 \oplus i_0 \oplus t_1 \in Tree_{\omega}(b_0)$. There is a leaf $i_1$ in this tree having the label $b$. We take the tree $(t_0 \oplus i_0 \oplus t_1) \oplus i_1 \oplus t_1$ and we repeat this step. Thus we obtain a sequence of trees:

$$
\begin{align*}
\alpha_0 &= t_0 \oplus i_0 \oplus t_1 \\
\alpha_{j+1} &= \alpha_j \oplus i_{j+1} \oplus t_1, \quad j \geq 0
\end{align*}
$$

such that each $\alpha_j$ contains a leaf labelled by $b$. If $B_0, B_1, \ldots$ are the corresponding sets of the nodes for these trees then $\text{Card}(B_0) < \text{Card}(B_1) < \ldots$. Let be $[t_g]$ the greatest element of $Tree_{\omega}(b_0)/_\approx$. If $A_g$ is the node set of $t_g$ then we have $\text{Card}(B_0) < \text{Card}(B_1) < \ldots < \text{Card}(A_g)$, which is not true since $A_g$ is a finite set.

2) $b = b_0$
We take $t_1 = t_0$ and we proceed as above. Thus, the assumption $b \in T(b)$ for some $b \in \{b_0\} \cup T(b_0)$ is not true.

Proposition 3.4 Let be $b_0 \in N_L$. Suppose $b \not\in T(b)$ for every $b \in \{b_0\} \cup T(b_0)$. We define recursively the following sets $Q_1, Q_2, \ldots$ as follows:

- $((l), (b)) \in Q_1$ iff $l \in \{1, \ldots, \omega_1(b_0)\}$ and $b = pr_1\omega_2(b_0)$
- $((l_1, \ldots, l_k, l), (b_1, \ldots, b_k, b)) \in Q_{k+1}$ if and only if $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Q_k$, $l \in \{1, \ldots, \omega_1(b_k)\}$, $b = pr_1\omega_2(b_k)$

Then the following properties are satisfied:

1) $Q_{n(l)+1} = Q_{n(l)+2} = \ldots = \emptyset$, where $n_l = \text{Card}(N_L)$
2) $X = \bigcup_{k=1}^{n(l)} Q_k$ satisfies the $\omega - b_0$-conditions
3) For every $Y$ satisfying the $\omega - b_0$-conditions we have $Y \subseteq X$

Proof. For every $k \geq 1$, if $((l_1, \ldots, l_k), (b_1, \ldots, b_k)) \in Q_k$ then $b_j \in T(b_i)$ for every $i, j$ such that $0 \leq i < j \leq k$. We prove this assertion by induction on $k$. For $k = 1$, if $((l), (b)) \in Q_1$ then $b = pr_1\omega_2(b_0)$ and $l \in \{1, \ldots, \omega_1(b_0)\}$. Then $b \in U(\{b_0\}) \subseteq T(b_0)$. Assuming the assertion is true for $k$, if $((l_1, \ldots, l_k, l), (b_1, \ldots, b_k, b)) \in Q_{k+1}$ then $b_j \in T(b_i)$ for $0 \leq i < j \leq k$ and $b = pr_1\omega_2(b_k)$, where $l \in \{1, \ldots, \omega_1(b_k)\}$. Then $b \in U(\{b_k\}) \subseteq T(b_k)$. By inductive assumption $b_k \in T(b_i)$ for every $i \in \{0, \ldots, k-1\}$. By Proposition 3.2 it follows that $b \in T(b_i)$ for each $i \in \{0, \ldots, k-1\}$. Thus, if $((l_1, \ldots, l_k, l), (b_1, \ldots, b_k, b_{k+1})) \in Q_{k+1}$ then $T(b_i) \neq \emptyset$ for every $i \in \{1, \ldots, k\}$, therefore $b_1, \ldots, b_k \in N_L$. The sequence $b_0, b_1, \ldots, b_k$ satisfies the condition $b_i \neq b_j$ for $i \neq j$. Really, if $b_i = b_j = b$ for some $i < j$ then we have $b \in T(b)$, where $b \in \{b_0\} \cup T(b_0)$, which is not true. Therefore, if $Q_{k+1} \neq \emptyset$ then $k + 1 \leq n_l$. If we...
suppose that $Q_{n(t)+1} \neq \emptyset$ then there is $((k_1, \ldots , k_{n(t)+1}), (b_1, \ldots , b_{n(t)+1})) \in Q_{n(t)+1}$. This implies that $b_0, b_1, \ldots , b_{n(t)}$ are distinct elements of $N_L$, which is not true. Thus the first property is proved.

Obviously $X = \bigcup_{k=1}^{n(t)} Q_k$ satisfies the $\omega - b_0$-conditions. Let $Y$ be a set satisfying the $\omega - b_0$-conditions. We verify by induction that for every $k \geq 1$, if $((l_1, \ldots , l_k), (b_1, \ldots , b_k)) \in Y$ then $((l_1, \ldots , l_k), (b_1, \ldots , b_k)) \in Q_k$. For $k = 1$ the assertion is true. We assume the assertion is true for $k$ and let be $((l_1, \ldots , l_k, t), (b_1, \ldots , b_k, b)) \in Y$. Since $Y$ satisfies the $\omega - b_0$-conditions we have $((l_1, \ldots , l_k), (b_1, \ldots , b_k)) \in Y$, $t \in \{1, \ldots , \omega_1(b_k)\}$, $b = pr_{\omega_2}(b)$ and $((l_1, \ldots , l_k, m), (b_1, \ldots , b_k, pr_{m}(\omega_2(b))) \in Y$ for $m \in \{1, \ldots , \omega_1(b_k)\}$. By inductive assumption we have $((l_1, \ldots , l_k), (b_1, \ldots , b_k)) \in Q_k$ and by the definition of $Q_{k+1}$ we have $((l_1, \ldots , l_k, m), (b_1, \ldots , b_k, pr_{m}(\omega_2(b))) \in Q_{k+1}$ for every $m \in \{1, \ldots , \omega_1(b_k)\}$. Particularly we have $((l_1, \ldots , l_k, t), (b_1, \ldots , b_k, b)) \in Q_{k+1}$. Thus $Y \subseteq \bigcup_{k=1}^{n(t)} Q_k = X$.

**Proposition 3.5** Suppose that $b_0 \in N_L$ and $b \notin T(b)$ for every $b \in \{b_0\} \cup T(b_0)$. Then the lattice $(\text{Tree}_\omega(b_0)/\approx, \lor, \land)$ contains a greatest element.

**Proof.** Let us consider $X = \bigcup_{k=1}^{n(t)} Q_k$ from Proposition 3.4. Since $X$ satisfies the $\omega - b_0$-conditions it follows that there is $t_0 \in \text{Tree}_\omega(b_0)$ such that $S(t_0) = X$. Let be now $[t] \in \text{Tree}_\omega(b_0)/\approx$. $S(t)$ satisfies the $\omega - b_0$-conditions, therefore $S(t) \subseteq S(t_0)$. This implies $t \leq t_0$, therefore $[t] \leq [t_0]$.

**Proposition 3.6** Suppose $(\text{Tree}_\omega(b_0)/\approx, \lor, \land)$ contains a greatest element. If $t_0 = (A_0, R_0, b_0) \in \text{Tree}_\omega(b_0)$ such that for every leaf $i$ of $t_0$ we have $h_0(i) \in T_L$ then for every $t \in \text{Tree}_\omega(b_0)$ we have $t \leq t_0$.

**Proof.** Suppose that for every leaf $i$ of $t_0$ we have $h_0(i) \in T_L$. Let be $t = (A, R, h) \in \text{Tree}_\omega(b_0)$. We shall verify that $S(t) \subseteq S(t_0)$, which will show that $t \leq t_0$. Let be $((l_1, \ldots , l_s), (b_1, \ldots , b_s)) \in S(t)$. There is $((1, i_1), (1, i_2)) \in \text{Path}(t)$ such that $(1, i_1) \in R^{(i_1)}$, $(1, i_2) \in R^{(i_2)}$, ..., $(i_{s-1}, i_s) \in R^{(i_s)}$, $h(i_1) = b_1$, ..., $h(i_s) = b_s$. Only the nodes labelled by nonterminal labels may have direct descendants, therefore $b_1, \ldots , b_s - 1 \in N_L$. In order to simplify the notation we denote $\omega_1(b_i) = n_j$ for $j \in \{0, \ldots , s-1\}$. For every $j \in \{1, \ldots , s\}$ there is $u_j = [(i_{j-1}, k_{j}^{(i)}) \ldots (i_{j-1}, k_{n_{j-1}}^{(i)})] \in R$ such that $i_j = \frac{k_{j}^{(i)}}{i_{j-1}}$, where $i_0 = 1$. Since $t_0$ is an $\omega$-labelled tree it follows that there is $v_i = [(1, r_{1}^{(i)}), \ldots , (1, r_{n_{i}}^{(i)})] \in R_0$ such that $h_0(r_{n_{i}}^{(i)}) = b_i$. We denote $j_1 = r_{1}^{(i)}$. By induction on $p$ we prove that for every $p \in \{1, \ldots , s\}$ there is $v_p = ([j_{p-1}, r_{1}^{(p)}], \ldots , (j_{p-1}, r_{n_{p-1}}^{(p)})) \in R_0$ such that $h_0(r_{n_{p-1}}^{(p)}) = b_p$, where $j_p = r_{p}^{(i)}$ and $j_0 = 1$. For $p = 1$ the property is verified. We assume the property is verified for some $p \in \{1, \ldots , s-1\}$. Since $b_p \in N_L$ and $h_0(j_p) = b_p$ it follows that $j_p$ has $n_p$ direct descendants in $t_0$. Thus there is $v_{p+1} = ([j_p, (r_{1}^{(p+1)}), \ldots , (j_p, r_{n_{p+1}}^{(p+1)}))] \in R_0$ such that $h_0(r_{n_{p+1}}^{(p+1)}) = b_{p+1}$. We take $j_{p+1} = r_{p+1}^{(i)}$. In this way we obtain $(1, j_1, \ldots , j_s) \in \text{Path}(t_0)$ such that $(1, j_1) \in R_0^{(i_1)}, (j_1, j_s) \in R_0^{(i_s)}, \ldots , (j_{s-1}, j_s) \in R_0^{(i_s)}$, $h_0(j_1) = b_1$, ..., $h_0(j_s) = b_s$. Therefore $((l_1, \ldots , l_s), (b_1, \ldots , b_s)) \in S(t_0)$.

**Corollary 3.1** Suppose $(\text{Tree}_\omega(b_0)/\approx, \lor, \land)$ contains a greatest element. If $t_1 = (A_1, R_1, h_1) \in \text{Tree}_\omega(b_0)$ and $t_2 = (A_2, R_2, h_2) \in \text{Tree}_\omega(b_0)$ are such that $h_1(i) \in T_L$ and $h_2(j) \in T_L$ for each leaves $i$ and $j$ then $t_1 \approx t_2$, therefore $[t_1] = [t_2]$. 

**LATTICES OF LABELLED ORDERED TREES -II**
Proof. Really, by Proposition 3.6 we have $t_1 \preceq t_2$ and $t_2 \preceq t_1$, therefore $t_1 \approx t_2$.

Remark 3.1 Suppose $(\text{Tree}_\omega(b_0)/\approx, \lor, \land)$ contains a greatest element. The tree $t_0 = (A_0, R_0, h_0) \in \text{Tree}_\omega(b_0)$ is a representative of the greatest element if and only if for every leaf $i$ of $t_0$ we have $h_0(i) \in T_L$.

References


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