Proving the existence of Labelled Stratified Graphs

Nicolae Ţăndăreanu
Faculty of Mathematics and Computer Science,
University of Craiova,
str.A.I.Cuza 13, 1100-Craiova, Romania
e-mail: ntand@oltenia.ro

Abstract

The concept of *labelled stratified graph* was used in [2] in order to obtain the concept of knowledge base with output. In what concerns the existence of such structure nothing is proved there. Using some concepts of universal algebra we prove in this paper that for each labelled graph G there is a labelled stratified graph $\mathcal{G}$ over G. We give a method to obtain such structures. The environment of $\mathcal{G}$ is covered by some set L of labels, which is divided into several layers. These layers are used by the inference process which can be realized by means of $\mathcal{G}$. Although the environment is a finite set, L may be an infinite one and in this case an infinite hierarchy of layers is obtained. Such a case is given in an example.

The research was supported by National Council for Scientific Research of Higher Education.

1 Introduction

The concept of *labelled stratified graph* was suggested by the graph-based methods for knowledge representation, especially by semantic networks. Various forms of semantic networks use labels as *inst, subc* to suggest the corresponding semantics: x is an instance of y, every x is y ([3]). We generalize this process by considering abstract labels on arcs and using several concepts and results of universal algebra we obtain the concept of labelled stratified graph.

The concept of labelled stratified graph was applied to introduce the concept of knowledge base with output ([2]). As is stated in [2], a knowledge base with output is a collection of components, one of them being a labelled stratified graph. Afterwards I found that a labelled stratified graph can be used independently, that is, without using the other components of a knowledge base with output. On the other hand nothing is proved in [2] concerning the existence of such structure. In this paper we deal with this problem.

Generally speaking a labelled stratified graph can be used in automated reasoning. The inference is realized by means of an environment. If we denote by G a labelled graph then an environment for G is a finite set of binary relations which is closed under the product operation. We show that for each environment there is a labelled stratified graph $\mathcal{G}$ over G. We give also a method to obtain such structures. The environment is a component of $\mathcal{G}$ and it is covered by a set L of labels. The set L is divided into several layers. An inference process can be realized in $\mathcal{G}$ and in order to realize this process the layers of L are used. An example of such inference is given in [2]. Although the environment of $\mathcal{G}$ is a finite set, the set L may be an infinite one and this property is relieved by means of an example given in this paper.

The structure of this paper is the following: in Section 2 we present several concepts and notations, which are useful in the remainder of the paper; in Section 3 we develop the concept of labelled stratified graph and we prove that for every labelled graph G and each environment there is a labelled stratified graph; we give a method by means of which we obtain such a structure.

2 General concepts of universal algebra

We consider a non empty set A; $B \subseteq A$ means that B is a subset of A; the empty set $\emptyset$ is a subset of every set; by 2^A we denote the power set of A, that is the set of all subsets of A.

By a *binary partial operation* on A we understand a partial mapping f from $A \times A$ to A. This means that f is defined for the elements of some set $dom(f)$, where $dom(f) \subset A \times A$. We shall use the notation $f : dom(f) \longrightarrow A$. In the case when $dom(f) = A \times A$ we say that f is a *binary operation* on A.

We shall write $f \prec g$ if $f : dom(f) \longrightarrow A$ and $g : dom(g) \longrightarrow A$ are two functions such that $dom(f) \subseteq dom(g)$ and $f(x) = g(x)$ for all $x \in dom(f)$.

By a *partial σ-algebra* we understand a pair $\mathcal{A}=(A, \sigma_A)$, where A is the support set of $\mathcal{A}$ and σ_A is *a partial binary operation* on A. If $dom(\sigma_A) = A \times A$ then we say that $\mathcal{A}$ is a *σ-algebra.*

Let $\mathcal{A}=(A, \sigma_A)$ be a partial σ-algebra. A subset $B \subseteq A$ is a *closed set* in $\mathcal{A}$ if the following condition is fulfilled: if $(x_1, x_2) \in dom(\sigma_A) \cap (B \times B)$ then $\sigma_A(x_1, x_2) \in B$. If $B \subseteq A$ then *the closure of B in $\mathcal{A}$* is the smallest closed set containing B. The closure of B is denoted by $\overline{B}$ and obviously if B is a closed set then $\overline{B} = B$. It can be shown that if B is not a closed set then $\overline{B} = \bigcup_{n \geq 0} B_n$ where

$$\begin{cases} B_0 = B \\ B_{n+1} = B \cup \{\sigma_A(x_1, x_2) \mid (x_1, x_2) \in dom(\sigma_A) \cap (B_n \times B_n)\}, n \geq 0 \end{cases}$$

We consider now the partial σ-algebras $\mathcal{A}=(A, \sigma_A)$ and $\mathcal{B}=(B, \sigma_B)$. The mapping $h : A \longrightarrow B$ is a *morphism of partial algebras* from $\mathcal{A}$ to $\mathcal{B}$ if for every $(x_1, x_2) \in dom(\sigma_A)$ the following conditions are fulfilled:

- $(h(x_1), h(x_2)) \in dom(\sigma_B)$
- $\sigma_B(h(x_1), h(x_2)) = h(\sigma_A(x_1, x_2))$

We shall use the notation $h : \mathcal{A} \longrightarrow \mathcal{B}$ to specify that h is a morphism from $\mathcal{A}$ to $\mathcal{B}$. A bijective morphism is an *isomorphism.* Two partial σ- algebras $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic algebras* if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Let $\mathcal{A}=(A, \sigma_A)$ be a σ-algebra and $M \subseteq A$. By definition, $\mathcal{A}$ is a *Peano σ- algebra over M* if the following conditions are fulfilled ([1]):

- $\overline{M} = A$
- $\sigma_A(x_1, x_2) \notin M$ for every $x_1, x_2 \in A$
- for every $x_1, x_2, y_1, y_2 \in A$, from $\sigma_A(x_1, x_2) = \sigma_A(y_1, y_2)$ we deduce $x_1 = y_1$ and $x_2 = y_2$

By definition, the σ-algebra $\mathcal{A}=(A,\sigma_A)$ is *free-generated* by $M \subseteq A$ if for every σ-algebra $\mathcal{B}=(B,\sigma_B)$ and every function $f: M \longrightarrow B$ there exists a morphism and only one, $h: A \longrightarrow B$, such that $f \prec h$.

For every set M there is a Peano σ-algebra over M. In order to obtain such an algebra we proceed as follows. We may assume $\sigma \notin M$. We take the σ-algebra $\mathcal{H}=(H,\sigma_H)$, where H is the set of all the nonempty words over $\{\sigma\} \cup M$ and $\sigma_H(x_1,x_2) = \sigma x_1 x_2$. The pair $\mathcal{A}=(A,\sigma_A)$, where A is the closure of M in $\mathcal{H}$ and σ_A is the restriction of σ_H on A, is a Peano σ-algebra over M. Two Peano σ-algebras over the same set M are isomorphic algebras (particularly they are isomorphic with $\mathcal{A}$) because a Peano σ- algebra over M is a σ-algebra free generated by M and two σ-algebras free generated by M are isomorphic algebras ([1]). Thus, if $M = \{a,b\}$ then the set $H = \{a, b, \sigma(a,a), \sigma(b,b), \sigma(a,b), \ldots\}$ gives the Peano σ-algebra over M. The elements of H are called *terms* by some authors. We observe that the elements of H are nonempty *strings* over M.

Let $\mathcal{A}=(A,\sigma_A)$ be a partial σ-algebra. We denote by *Initial($\mathcal{A}$)* the set of all the subsets $B \subseteq A$ satisfying the following condition: for every $(x_1,x_2) \in dom(\sigma_A)$, from $\sigma_A(x_1,x_2) \in B$ we deduce $x_1, x_2 \in B$. We observe now that if $\mathcal{A}=(A,\sigma_A)$ is a Peano σ-algebra over M and $B \in Initial(\mathcal{A})$ is such that $M \subseteq B$ then denoting by σ_B the restriction of σ_A to B, we obtain a partial σ-algebra $\mathcal{B}=(B,\sigma_B)$. In this case the closure of M in $\mathcal{B}$ is B.

Let us consider a finite set S. A *binary relation* over S is a subset $\rho \subseteq S \times S$. If $\rho_1 \in 2^{S\times S}$ and $\rho_2 \in 2^{S\times S}$ then we define: $\rho_1 \circ \rho_2 = \{(x,y) \in S \times S \mid \exists z \in S : (x,z) \in \rho_1, (z,y) \in \rho_2\}$. We define the mapping $prod : dom(prod) \longrightarrow 2^{S\times S}$ as follows:

$$dom(prod) = \{(\rho_1,\rho_2) \in 2^{S\times S} \times 2^{S\times S} \mid \rho_1 \circ \rho_2 \neq \emptyset\}$$

$$prod(\rho_1,\rho_2) = \rho_1 \circ \rho_2$$

The mapping $prod$ is called the *product operation*. The pair $(2^{S\times S}, \sigma_S)$, where $\sigma_S = prod$ becomes a partial σ- algebra.

By $Card(X)$ we denote the cardinal number of the set X.

3 Labelled stratified graphs

Let S be a finite set. We denote by $R_S(prod)$ the following set of partial mappings:

$$R_S(prod) = \{h \mid h \prec prod, dom(h) \neq \emptyset\}$$

By σ_S we denote an arbitrary element of $R_S(prod)$. In particular we may have $\sigma_S = prod$. For a given $\sigma_S \in R_S(prod)$ we consider the partial σ-algebra $\mathcal{S} = (2^{S\times S}, \sigma_S)$. If T_0 is a nonempty subset of $2^{S\times S}$ then we denote by $Cl_{\sigma_S}(T_0)$ the closure of T_0 in the partial σ-algebra $\mathcal{S} = (2^{S\times S}, \sigma_S)$.

Proposition 3.1 *For every $\sigma_S \in R_S(prod)$ we have $T_0 \subseteq Cl_{\sigma_S}(T_0) \subseteq Cl_{prod}(T_0)$.*

Proof. We have $T_0 \subseteq Cl_{\sigma_S}(T_0)$ by definition of the closure $Cl_{\sigma_S}(T_0)$. Because $\sigma_S \prec prod$ it follows that every closed set under $prod$ is a closed set under σ_S. Therefore $Cl_{prod}(T_0)$ is closed under σ_S. But $T_0 \subseteq Cl_{prod}(T_0)$ and $Cl_{\sigma_S}(T_0)$ is the smallest closed set under σ_S containing T_0. Thus $Cl_{\sigma_S}(T_0) \subseteq Cl_{prod}(T_0)$. ■

Proposition 3.2 *Let be $T_0 \subseteq 2^{S\times S}$ such that $\emptyset \notin T_0$. Let $\mathcal{S} = (2^{S\times S}, \sigma_S)$ be a partial σ-algebra such that $\sigma_S \in R_S(prod)$. The sequence $\{X_n\}_n$ defined by:*

$$\begin{cases} X_0 = T_0 \\ X_{n+1} = T_0 \cup \{d \mid \exists(d_1, d_2) \in (X_n \times X_n) \cap dom(\sigma_S) : d = \sigma_S(d_1, d_2)\}, \qquad n \geq 0 \end{cases}$$

satisfies the following properties:

i) $X_n \subseteq X_{n+1}$ *for any* $n \geq 0$

ii) If $X_n = X_{n+1}$ *then* $X_n = X_{n+p}$ *for any* $p \geq 1$

iii) Let be $k = Card(S)$ *and* $t = 2^{k^2}$. *There is* $n \leq t$ *such that* $X_n = X_{n+1}$

iv) $Cl_{\sigma_S}(T_0) = X_{n(T_0)}$ *where* $n(T_0)$ *is the smallest* n *satisfying the previous condition*

Proof. We observe first that for every n we have:

$$X_{n+1} = X_n \cup \{d \mid \exists(d_1, d_2) \in (X_n \times X_n) \cap dom(\sigma_S) : d = \sigma_S(d_1, d_2)\}, n \geq 0$$

There is 2^{k^2} binary relations over S, therefore we have $iii)$. The last property is obtained using the previous properties and taking into account the relation $Cl_{\sigma_S}(T_0) = \bigcup_{n\geq 0} X_n$. ∎

Definition 3.1 *Let* $T_0 \subseteq 2^{S\times S}$ *such that* $T_0 \neq Cl_{prod}(T_0)$. *We denote*

$$Env(T_0) = \{(T, \sigma_S) \mid T \subseteq 2^{S\times S}, \sigma_S \in R_S(prod), T = Cl_{\sigma_S}(T_0)\}$$

An element of $Env(T_0)$ *is called an* **environment** *for* T_0.

An environment may be considered as a partial σ-algebra. Thus, if $(T, \sigma_S) \in Env(T_0)$ and define

$$\begin{cases} dom(\sigma_T) = (T \times T) \cap dom(\sigma_S) \\ \sigma_T(x, y) = \sigma_S(x, y) \text{ for } (x, y) \in dom(\sigma_T) \end{cases}$$

then $\mathcal{A} = (T, \sigma_T)$ is a partial σ-algebra. For this reason we use also the notation $(T, \sigma_T) \in Env(T_0)$.

In order to emphasize this aspect we consider the following example:

$S = \{x_1, x_2, x_3, x_4\}$

$\rho_1 = \{(x_1, x_2), (x_2, x_1)\}$, $\rho_2 = \{(x_1, x_3), (x_2, x_3)\}$, $\rho_3 = \{(x_3, x_4)\}$,
$\rho_4 = \{(x_1, x_1), (x_2, x_2)\}$, $\rho_5 = \{(x_1, x_4), (x_2, x_4)\}$

$\sigma_S : 2^{S\times S} \times 2^{S\times S} \longrightarrow 2^{S\times S}$; $dom(\sigma_S) = \{(\rho_1, \rho_1), (\rho_1, \rho_2), (\rho_2, \rho_3), (\rho_1, \rho_4), (\rho_4, \rho_2)\}$

$\sigma_S(\rho_1, \rho_1) = \rho_4$, $\sigma_S(\rho_1, \rho_2) = \rho_2$, $\sigma_S(\rho_2, \rho_3) = \rho_5$, $\sigma_S(\rho_1, \rho_4) = \rho_1$, $\sigma_S(\rho_4, \rho_2) = \rho_2$

Taking $T_0 = \{\rho_1, \rho_2\}$ and $T = \{\rho_1, \rho_2, \rho_4\}$ we have $Cl_{\sigma_S}(T_0) = T$. Really, $T_1 = T_0 \cup \{\rho_4\}$, $T_2 = T_1 = T$. Thus, $dom(\sigma_T) = (T \times T) \cap dom(\sigma_S) = \{(\rho_1, \rho_1), (\rho_1, \rho_2), (\rho_1, \rho_4), (\rho_4, \rho_2)\} \subset dom(\sigma_S)$ and $\sigma_T \prec \sigma_S$. We obtain the partial σ-algebra $\mathcal{A}_T = (\{\rho_1, \rho_2, \rho_4\}, \sigma_T)$. This aspect is stated in the next definition.

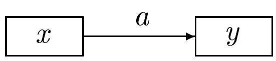

Figure 1: A labelled arc

Definition 3.2 *Let be* $(T, \sigma_S) \in Env(T_0)$. *The pair* $\mathcal{A}_T = (T, \sigma_T)$ *is called the partial* σ-*algebra* **associated with** (T, σ_S) *if* $dom(\sigma_T) = (T \times T) \cap dom(\sigma_S)$ *and* $\sigma_T(x,y) = \sigma_S(x,y)$ *for* $(x,y) \in dom(\sigma_T)$.

In a labelled graph both the nodes and the arcs are labelled entities. Moreover, some set of binary relations is relieved by such structure. The correspondence between these entities is specified in the following definition:

Definition 3.3 *We consider two finite sets* S *and* L_0 *such that* $S \cap L_0 = \emptyset$. *An element of* S *is called a* **node label**; *the elements of* L_0 *are called* **arc labels**. *Let* T_0 *be a set of binary relations on* S, *such that* $\emptyset \notin T_0$. *Let* $f_0 : L_0 \longrightarrow T_0$ *be a surjective function. The system* $G = (S, L_0, T_0, f_0)$ *is called a* **labelled graph**.

A labelled graph $G = (S, L_0, T_0, f_0)$ is represented as follows. Every node is represented by a rectangle containing its label. We draw a labelled arc $a \in L_0$ from the node $x \in S$ to the node $y \in S$ if and only if $(x,y) \in f_0(a)$ (Figure 1). Every element of S will designate a node of the graph and only one; thus some bijection between S and the set of the nodes can be established. We say that $a \in L_0$ is a label of the binary relation $f_0(a) \in T_0$. Because f_0 is a surjective function, it follows that every element of T_0 has at least one label.

Definition 3.4 *Let* L_0 *be a nonempty set. Let* $\mathcal{H} = (H, \sigma_H)$ *be the Peano* σ-*algebra over* L_0. *For every* $L \in Initial(\mathcal{H})$ *such that* $L \supseteq L_0$ *we define the partial* σ-*algebra* $\mathcal{A}_H(L) = (L, \sigma_L)$, *where:*

- $dom(\sigma_L) = \{(x,y) \in L \times L \mid \sigma_H(x,y) \in L\}$
- $\sigma_L(x,y) = \sigma_H(x,y)$ *for every* $(x,y) \in dom(\sigma_L)$

The pair $\mathcal{A}_H(L) = (L, \sigma_L)$ *is named* **the partial σ-algebra associated** *to* $L \in Initial(\mathcal{H})$.

For instance, if $H = \{a, b, \sigma_H(a,a), \sigma_H(a,b), \sigma_H(b,a), \ldots\}$ is the Peano σ-algebra over $\{a, b\}$ then

$$L = \{a, b, \sigma_H(a,a), \sigma_H(a,b), \sigma_H(b,a), \sigma_H(a, \sigma_H(a,b))\} \in Initial(\mathcal{H})$$

and

$$dom(\sigma_L) = \{(a,a), (a,b), (b,a), (a, \sigma_H(a,b))\}$$

Using the string notation we may write also $\sigma_L(a,a) = \sigma_H aa$, $\sigma_L(a,b) = \sigma_H ab$, $\sigma_L(b,a) = \sigma_H ba$, $\sigma_L(a, \sigma_L(a,a)) = \sigma_H a \sigma_H aa$.

Definition 3.5 *Let* $G = (S, L_0, T_0, f_0)$ *be a labelled graph. A* **labelled stratified graph over** G *is a system* $\mathcal{G} = (G, L, T, \sigma_T, f)$ *such that:*

- $(T, \sigma_T) \in Env(T_0)$
- $L \in Initial(\mathcal{H})$ *and* $L_0 \subseteq L$, *where* $\mathcal{H}$ *is the Peano* σ*-algebra over* L_0
- $f : \mathcal{A}_H(L) \longrightarrow \mathcal{A}_T$ *is a surjective morphism such that* $f_0 \prec f$ *and if* $(f(u), f(v)) \in dom(\sigma_T)$ *then* $\sigma_H(u, v) \in L$

Let us analyse this definition. We want to relieve some property of f, which is used later. Because σ is a symbol of arity 2, we have $dom(\sigma_L) \subseteq L \times L$ and $dom(\sigma_T) \subseteq T \times T$. The mapping $f : L \longrightarrow T$ is a morphism, that is, in the diagram

$$\begin{array}{ccc} L \times L & \xrightarrow{\sigma_L} & L \\ {\scriptstyle f \times f}\downarrow & & \downarrow{\scriptstyle f} \\ T \times T & \xrightarrow[\sigma_T]{} & T \end{array}$$

for every $(u, v) \in dom(\sigma_L)$ the following two properties are satisfied:

$(f(u), f(v)) \in dom(\sigma_T)$

$\sigma_T(f(u), f(v)) = f(\sigma_L(u, v))$

Thus we have

$$dom(\sigma_L) \subseteq \{(u, v) \in L \times L \mid (f(u), f(v)) \in dom(\sigma_T)\} \tag{1}$$

If f satisfies definition 3.5 then from $(f(u), f(v)) \in dom(\sigma_T)$ we have $\sigma_H(u, v) \in L$. By definition 3.4 we have $(u, v) \in dom(\sigma_L)$. Thus we have

$$\{(u, v) \in L \times L \mid (f(u), f(v)) \in dom(\sigma_T)\} \subseteq dom(\sigma_L) \tag{2}$$

From (1) and (2) it follows that if f satisfies definition 3.5 then f satisfies also the following condition:

$$dom(\sigma_L) = \{(u, v) \in L \times L \mid (f(u), f(v)) \in dom(\sigma_T)\}$$

Trying to explain why this concept is named labelled stratified graph we remark the following two facts:

(1) the environment (T, σ_T) gives two of the components of $\mathcal{G}$; by the surjective morphism f, the set T is covered by L; in other words every binary relation of T has at least one label and the labels are assigned by f; thus, the elements of T are labelled binary relations

(2) based on the next definition we observe that the set L of all the labels is divided into several layers; the first layer is given by L_0; each element of the layer i is obtained by means of two elements, one of them belonging to the layer $i-1$ and the other being in the set union of the layers $0, 1, \ldots, i-1$.

Definition 3.6 *Let $\mathcal{G} = (G, L, T, \sigma_T, f)$ be a labelled stratified graph. We define*

$$\begin{cases} Layer(L,0) = L_0 \\ Layer(L,n+1) = L \cap (H_{n+1} \setminus H_n), \ n \geq 0 \end{cases} \tag{3}$$

where

$$\begin{cases} H_0 = L_0 \\ H_{n+1} = H_n \cup \{\sigma_H(u,v) \mid u, v \in H_n\} \end{cases}$$

and $H = \bigcup_{n \geq 0} H_n$ is the Peano σ-algebra over L_0. The set $Layer(L,n)$ is called the n^{th} layer of L.

In what follows we shall prove that for every labelled graph $G = (S, L_0, T_0, f_0)$ and for every $(T, \sigma_T) \in Env(T_0)$ there is a labelled stratified graph over G, we shall give a method to obtain such a structure and we shall characterize its layers.

Let $G = (S, L_0, T_0, f_0)$ be a labelled graph such that $T_0 \neq Cl_{prod}(T_0)$. We observe that $dom(f_0) = H_0$ and we can define recursively for every natural number $n \geq 0$:

- $D_{n+1} = \{\sigma_H(u,v) \in H_{n+1} \setminus H_n \mid u, v \in dom(f_n), \quad (f_n(u), f_n(v)) \in dom(\sigma_T)\}$
- $dom(f_{n+1}) = dom(f_n) \cup D_{n+1}$
- $f_{n+1}(x) = \begin{cases} f_n(x) & \text{if} \quad x \in dom(f_n) \\ \sigma_T(f_n(u), f_n(v)) & \text{if} \quad x = \sigma_H(u,v) \in D_{n+1} \end{cases}$

As a consequence of these definitions the following properties are obtained:

(a) $D_i \cap D_j = \emptyset$ for $i \neq j$; $L_0 \cap D_i = \emptyset$ for every $i \geq 1$

(b) $dom(f_n) = H_0 \cup \bigcup_{k=1}^{n} D_k \subseteq H_n$, for every $n \geq 0$

(c) $dom(f_n) \cap D_{n+1} = \emptyset$ for every $n \geq 0$

Taking into account these properties we obtain the following proposition:

Proposition 3.3 *For every $n \geq 0$ the following properties are true:*

1) *the function f_n is well defined*

2) $f_n \prec f_{n+1}$

3) $f_n : dom(f_n) \longrightarrow T$

Proof. Because H is a Peano σ-algebra it follows that if $x \in H_{n+1} \setminus H_n$ then we can write $x = \sigma_H(u,v)$ for some $u, v \in H_n$ only in one manner. Taking into account the definition of f_{n+1} and the fact that $dom(f_n) \cap D_{n+1} = \emptyset$, we deduce that the mapping f_{n+1} is well defined for every $n \geq 0$. Obviously the other two sentences are true. ■

Definition 3.7 *We define the mapping $f^* : dom(f^*) \longrightarrow T$ as follows:*

$$dom(f^*) = \bigcup_{n\in N} dom(f_n) = L_0 \cup \bigcup_{k\geq 1} D_k$$

$$f^*(x) = \begin{cases} f_0(x) & \text{if} \quad x \in L_0 \\ f_k(x) & \text{if} \quad x \in D_k \end{cases}$$

Proposition 3.4 *For every $n \geq 0$ we have $dom(f^*) \cap H_n = dom(f_n)$.*

Proof. Really, $dom(f^*) = dom(f_n) \cup \bigcup_{k\geq n+1} D_k$. It follows that $dom(f^*) \cap H_n = (dom(f_n) \cap H_n) \cup \bigcup_{k\geq n+1}(D_k \cap H_n) = dom(f_n)$ because $dom(f_n) \subseteq H_n$ and for $k \geq n+1$ we have $D_k \cap H_n = \emptyset$.

Proposition 3.5 *Let be $L^* = dom(f^*)$. The set L^* satisfies the following properties:*

$(i1)$ $L^* \supseteq L_0$

$(i2)$ $\sigma_H(u,v) \in L^*$ iff $\{u, v \in L^*$ and $(f^*(u), f^*(v)) \in dom(\sigma_T)\}$

$(i3)$ $L^* \in Initial(\mathcal{H})$

Proof. Because $(i1)$ is obtained directly from the definition of L^*, we shall prove $(i2)$.
We suppose $\sigma_H(u,v) \in L^* = dom(f^*)$. Let n be the smallest natural number such that $\sigma_H(u,v) \in dom(f_n)$. We have $n \geq 1$ since $\sigma_H(u,v) \notin L_0 = dom(f_0)$. It follows that $\sigma_H(u,v) \in dom(f_n) \setminus dom(f_{n-1}) = D_n$. By the definition of D_n it follows that $u, v \in dom(f_{n-1}) \subseteq L^*$ and $(f^*(u), f^*(v)) = (f_{n-1}(u), f_{n-1}(v)) \in dom(\sigma_T)$.
Conversely, let $u, v \in L^*$ such that $(f^*(u), f^*(v)) \in dom(\sigma_T)$. Because $L^* = L_0 \cup \bigcup_{n\geq 1} D_n$ we deduce that there are $p, q \in N$ such that $u \in D_p$ and $v \in D_q$. Taking $k = max\{p, q\}$ we obtain $u \in dom(f_k)$, $v \in dom(f_k)$ and $\sigma_H(u,v) \in H_{k+1} \setminus H_k$. It follows that $(f^*(u), f^*(v)) = (f_p(u), f_q(v)) = (f_k(u), f_k(v)) \in dom(\sigma_T)$, therefore $\sigma_H(u,v) \in D_{k+1} \subseteq L^*$. Thus $(i2)$ is true. In order to prove $(i3)$ we observe that if $\sigma_H(u,v) \in L^*$ then $u, v \in L^*$. ■

Based on proposition 3.5 we can consider the partial σ-algebra $\mathcal{A}_{L^*} = (L^*, \sigma_{L^*})$. On the other hand the same proposition permits to obtain the following property for σ_{L^*}, which is used later:

Corollary 3.1 $dom(\sigma_{L^*}) = \{(x,y) \in L^* \times L^* \mid (f^*(x), f^*(y)) \in dom(\sigma_T)\}$

Proof. If $(x,y) \in L^* \times L^*$ and $(f^*(x), f^*(y)) \in dom(\sigma_T)\}$ then by proposition 3.5 we have $\sigma_H(x,y) \in L^*$. By definition 3.4 it follows that $(x,y) \in dom(\sigma_{L^*})$. Conversely, if $(x,y) \in dom(\sigma_{L^*})$ then $(x,y) \in L^* \times L^*$ and $\sigma_H(x,y) \in L^*$. By proposition 3.5 we have $(f^*(x), f^*(y)) \in dom(\sigma_T)$. ■

Because $(T, \sigma_T) \in Env(T_0)$, applying proposition 3.2 we obtain a natural number $n(T_0)$ and the hierarchy $T_0 \subset T_1 \subset \ldots \subset T_{n(T_0)} = T_{n(T_0)+1} = \ldots = T$. We consider the following sets:

$$\begin{cases} B_0 = T_0 \\ B_i = T_i \setminus T_{i-1}, \qquad i \in \{1, \ldots, n(T_0)\} \end{cases}$$

We have $B_i \cap B_j = \emptyset$ for $i \neq j$ and $T_i = \bigcup_{j=0}^{i} B_j$ for $i \in \{0, \ldots, n(T_0)\}$.

Proposition 3.6 *For every* $i \in \{1, \ldots, n(T_0)\}$, $d \in B_i$ *if and only if*

i) there exist $u \in B_{i-1}$, $v \in T_{i-1}$ *such that* $d = \sigma_T(u, v)$ *or* $d = \sigma_T(v, u)$

ii) $d \neq \sigma_T(d_1, d_2)$ *for every* $d_1, d_2 \in T_{i-2}$

Proof We suppose that $d \in B_i = T_i \setminus T_{i-1}$ for some $i \in \{1, \ldots, n(T_0)\}$. There exist $u, v \in T_{i-1}$ such that $(u, v) \in dom(\sigma_T)$ and $d = \sigma_T(u, v)$. Two cases are possible:

a) $u, v \in T_{i-2}$; in this case we have $d \in T_{i-1}$, which is not true

b) $u \in B_{i-1}$ or $v \in B_{i-1}$, therefore *i)* is true

In order to prove *ii)* we suppose by contrary that $d = \sigma_T(d_1, d_2)$ for some $d_1, d_2 \in T_{i-2}$. Thus we have $d \in T_{i-1}$, which is not true.
Conversely, if $u \in B_{i-1}$ and $v \in T_{i-1}$ then $d = \sigma_T(u, v) \in T_i$. If $d \neq \sigma_T(d_1, d_2)$ for every $d_1, d_2 \in T_{i-2}$ then $d \notin T_{i-1}$. Therefore $d \in T_i \setminus T_{i-1} = B_i$. ■

Proposition 3.7 *For every* $i \in \{1, \ldots, n(T_0)\}$ *we have* $B_i \subseteq f_i(D_i) \subseteq T_i$

Proof. We denote $D_0 = L_0$. Obviously the proposition is true also for $i = 0$.
Firstly we prove by induction on i that $B_i \subseteq f_i(D_i)$ for every $i \in \{1, \ldots, n(T_0)\}$. We verify this property for $i = 1$. We have $D_1 = \{\sigma_H(u, v) \in H_1 \setminus H_0 \mid u, v \in dom(f_0), (f_0(u), f_0(v)) \in dom(\sigma_T)\}$. Let be $z \in B_1$. By proposition 3.6 it follows that $z = \sigma_T(d_1, d_2)$ for some $d_1, d_2 \in T_0$. Because $f_0 : L_0 \longrightarrow T_0$ is a surjective function, we deduce that there exist $a, b \in L_0$ such that $d_1 = f_0(a)$, $d_2 = f_0(b)$. Therefore we have $z = \sigma_T(f_0(a), f_0(b))$ for some $a, b \in H_0$. But
$\sigma_H(a, b) \in H_1 \setminus H_0$, $a, b \in dom(f_0)$, $(f_0(a), f_0(b)) \in dom(\sigma_T)$
therefore $\sigma_H(a, b) \in D_1$. We obtain $f_1(\sigma_H(a, b)) = \sigma_T(f_0(a), f_0(b)) = z$, therefore $B_1 \subseteq f_1(D_1)$.
We suppose now that $B_j \subseteq f_j(D_j)$ for $j \in \{1, \ldots, i\}$ and we shall prove that $B_{i+1} \subseteq f_{i+1}(D_{i+1})$. We consider an arbitrary element $z \in B_{i+1}$. By proposition 3.6 it follows that there exist $d_1 \in B_i$ and $d_2 \in B_j$ for some $j \in \{0, \ldots, i\}$ such that $z = \sigma_T(d_1, d_2)$ or $z = \sigma_T(d_2, d_1)$. Obviously it is enough to consider the situation when $z = \sigma_T(d_1, d_2)$. By the inductive assumption we have $B_i \subseteq f_i(D_i)$ and $B_j \subseteq f_j(D_j)$, therefore there exist $a \in D_i$, $b \in D_j$ such that $d_1 = f_i(a)$, $d_2 = f_j(b)$. It follows that $\sigma_H(a, b) \in H_{i+1} \setminus H_i$, $a \in dom(f_i)$, $b \in dom(f_j) \subseteq dom(f_i)$ and $(f_i(a), f_i(b)) = (f_i(a), f_j(b)) \in dom(\sigma_T)$. Thus $\sigma_H(a, b) \in D_{i+1}$ and $f_{i+1}(\sigma_H(a, b)) = \sigma_T(f_i(a), f_i(b)) = \sigma_T(f_i(a), f_j(b)) = \sigma_T(d_1, d_2) = z$.
We prove now that $f_i(D_i) \subseteq T_i$. If $z \in f_1(D_1)$ then $z = f_1(\sigma_H(u, v)) = \sigma_T(f_0(u), f_0(v))$ for some $u, v \in L_0$. Because $(f_0(u), f_0(v)) \in (B_0 \times B_0) \cap dom(\sigma_T)$, it follows that $z \in T_1$. We suppose that $f_j(D_j) \subseteq T_j$ for every $j < i$ and let be $z \in f_i(D_i)$. It follows that $z = f_i(\sigma_H(u, v)) =$

$\sigma_T(f_{i-1}(u), f_{i-1}(v))$ for some $u, v \in dom(f_{i-1})$ such that $(f_{i-1}(u), f_{i-1}(v)) \in dom(\sigma_T)$. There exist $p, q \in \{0, \ldots, i-1\}$ such that $u \in D_p$ and $v \in D_q$. We have $f_{i-1}(u) = f_p(u)$ and $f_{i-1}(v) = f_q(v)$. By the inductive assumption we have $f_p(u) \in T_p \subseteq T_{i-1}$ and $f_q(u) \in T_q \subseteq T_{i-1}$, Thus $z \in T_i$. ■

Corollary 3.2 *For every $i \in \{1, \ldots, n(T_0)\}$ we have $D_i \neq \emptyset$.*

Proof. We apply proposition 3.7. If by contrary, we suppose that for some $i \in \{1, \ldots, n(T_0)\}$ we have $D_i = \emptyset$ then $f_i(D_i) = \emptyset \supseteq B_i$. Thus $B_i = \emptyset$, which is not true. ■

Proposition 3.8 *Let $\mathcal{A}_H(L^*) = (L^*, \sigma_{L^*})$ be the partial Σ-algebra associated with L^*, where $L^* = dom(f^*)$. The mapping $f^* : \mathcal{A}_H(L^*) \longrightarrow \mathcal{A}_T$ given in definition 3.7 is a surjective morphism of σ-algebras.*

Proof. By *Corollary 3.1* we have $dom(\sigma_{L^*}) = \{(x, y) \in L^* \times L^* \mid (f^*(x), f^*(y)) \in dom(\sigma_T)\}$. We consider an arbitrary pair $(x_1, x_2) \in dom(\sigma_{L^*})$, therefore $(x_1, x_2) \in L^* \times L^* = dom(f^*) \times dom(f^*)$ and $(f^*(x_1), f^*(x_2)) \in dom(\sigma_T)$. Obviously $f^*(\sigma_{L^*}(x_1, x_2)) = f^*(\sigma_H(x_1, x_2)) = \sigma_T(f^*(x_1), f^*(x_2))$, therefore f^* is a morphism. Taking into account the facts $dom(f^*) = L_0 \cup \bigcup_{k \geq 1} D_k$, $T = \bigcup_{k=0}^{n(T_0)} B_k$ and using proposition 3.7, we deduce that f^* is a surjective function. ■

We can obtain now the following theorem:

Theorem 3.1 *If $G = (S, L_0, T_0, f_0)$ be a labelled graph and $(T, \sigma_T) \in Env(T_0)$ then $\mathcal{G}^* = (G, L^*, T, \sigma_T, f^*)$ is a labelled stratified graph over G*

Proof. Use proposition 3.5 and proposition 3.8. ■

The next proposition shows that the layers of $\mathcal{G}^*$ are exactly the sets D_n.

Proposition 3.9 *Let $\mathcal{G}^* = (G, L^*, T, \sigma_T, f^*)$ be the labelled stratified graph obtained in theorem 3.1. Then $Layer(L^*, n) = D_n$ for every $n \geq 0$.*

Proof. By definition 3.7 we have $L^* = \bigcup_{j \geq 0} D_j$. For every $j \geq 0$ we have $D_j \subseteq H_j \setminus H_{j-1}$, where for $j = 0$ we consider $H_{j-1} = H_{-1} = \emptyset$. It follows that

$$D_j \cap (H_{k+1} \setminus H_k) = \begin{cases} \emptyset \text{ for } j \neq k+1 \\ D_{k+1} \text{ for } j = k+1 \end{cases} \tag{4}$$

From (3) and (4) we obtain:

$$Layer(L^*, k+1) = L^* \cap (H_{k+1} \setminus H_k) = \bigcup_{j \geq 0} D_j \cap (H_{k+1} \setminus H_k) = D_{k+1}$$

For $n = 0$ the property is obviously true and thus the proposition is proved. ■

The set L^* may be divided into an infinite number of layers. In order to emphasize this fact we take the following example. We consider the labelled graph from Figure 2. We take $S = \{x_1, x_2, x_3\}$ and $L_0 = \{a, b\}$. We consider the binary relations

$$\rho_1 = \{(x_1, x_2), (x_2, x_1)\}, \;\; \rho_2 = \{(x_1, x_3), (x_2, x_3)\}$$

Let be $T_0 = \{\rho_1, \rho_2\}$, $T = Cl_{prod}(T_0)$ and $\sigma_T = prod$. If we denote $\rho_3 = \{(x_1, x_1), (x_2, x_2)\}$ then we obtain

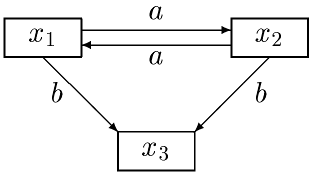

Figure 2: A labelled graph having an infinite hierarchy of layers

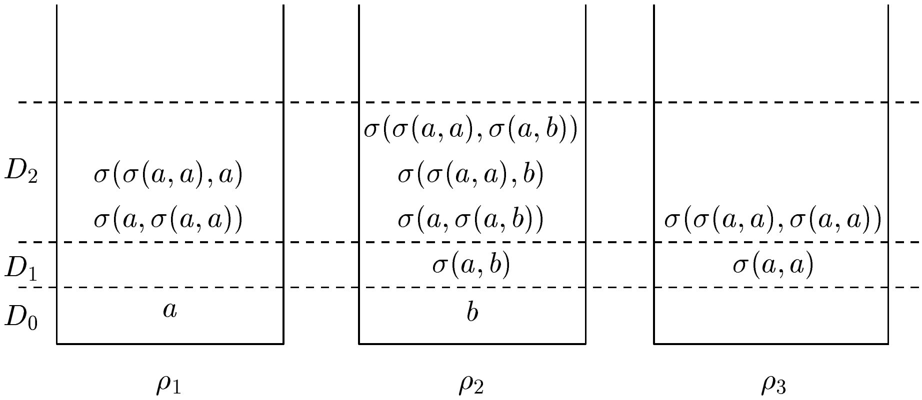

Figure 3: An infinite hierarchy of layers

$$\sigma_T(\rho_1,\rho_1)=\rho_3,\ \sigma_T(\rho_1,\rho_2)=\rho_2,\ \sigma_T(\rho_1,\rho_3)=\rho_1$$

$$\sigma_T(\rho_3,\rho_1)=\rho_1,\ \sigma_T(\rho_3,\rho_2)=\rho_2\ \sigma_T(\rho_3,\rho_3)=\rho_3$$

Applying proposition 3.2 we have $T_1=T_0\cup\{\rho_3\}$ and $T_2=T_1$ therefore $T=\{\rho_1,\rho_2,\rho_3\}$.
The computation of the elements D_n will conduce us to the elements specified in Figure 3. Taking into consideration the values of the mapping f^* we obtain three containers of labels, each of them containing all the labels for ρ_1, ρ_2, ρ_3 respectively. Each container contains an infinite set of labels. In order to verify this fact we denote

$$\sigma(P,Q)=\{\sigma(u,v)\mid u\in P, v\in Q\}$$

and for each natural number n we take

$$\sigma_n(A,B)=\bigcup_{j\leq n}[\sigma(A_n,B_j)\cup\sigma(A_j,B_n)]$$

where A_j, B_j are subsets of L, A is the sequence $A_0,A_1,\ldots$ and B is the sequence $B_0,B_1,\ldots$.
For every $j\geq 0$ and $i\in\{1,2,3\}$ we denote $D_j(\rho_i)=\{u\in D_j\mid f(u)=\rho_i\}$ and let $D(\rho_i)$ be the sequence $D_0(\rho_i),D_1(\rho_i),\ldots$.
Taking into account the manner in which σ_T is defined we obtain the following equations:

$$\begin{cases} D_{n+1}(\rho_3)=\sigma_n(D(\rho_1),D(\rho_1))\cup\sigma_n(D(\rho_3),D(\rho_3)) \\ D_{n+1}(\rho_2)=\sigma_n(D(\rho_1),D(\rho_2))\cup\sigma_n(D(\rho_3),D(\rho_2)) \\ D_{n+1}(\rho_1)=\sigma_n(D(\rho_1),D(\rho_3))\cup\sigma_n(D(\rho_3),D(\rho_1)) \end{cases} \quad (5)$$

We observe that $D_2(\rho_1), D_2(\rho_2)$ and $D_2(\rho_3)$ are nonempty sets. Based on (5) we can verify by induction that $D_n(\rho_1)$, $D_n(\rho_2)$ and $D_n(\rho_3)$ are also nonempty sets for every $n \geq 3$. Thus we obtain an infinite hierarchy of layers for L^*.

4 Conclusions and future work

Using some concepts and results from universal algebra we develop in this paper the concept of labelled stratified graph. We show that for each labelled graph G and each environment there is a labelled stratified graph $\mathcal{G}$ over G. We give also a method to obtain such structures. The environment is covered by a label set by means of a surjective morphism of universal algebra. This set is divided into several levels. In an example we show that may be an infinite number of layers and I hope this case is interesting in image synthesis. The concept of labelled stratified graph was used in order to introduce the concept of knowledge base with output ([2]), but it can be used independently in domains such as problem solving and expert systems. In what concerns the applications of the labelled stratified graphs in problem solving we relieve that this concept gives us a mathematical tool by which we can find all the solutions of a problem. We intend to use also this concept in image synthesis. In a forthcoming paper we study the algebra of all the labelled stratified graphs over G, where G is a given labelled graph.

References

[1] Boicescu, V., Filipoiu, A., Georgescu, G. and Rudeanu, S., "Lukasiewicz-Moisil Algebras", Annals of Discrete Mathematics **49**, North-Holland, 1991.

[2] Ţăndăreanu, N., Knowledge Bases with Output, Knowledge and Information Systems, 2 (2000), 4, 438-460

[3] Way, E.C., "Knowledge Representation and Metaphor", Kluwer Academic Publisher, Studies in cognitive systems, Vol. **7**, 1991.