An Overview of Stratified Graphs and Their Applications

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ABSTRACT

In this paper a synthesis of the knowledge representation method based on stratified graphs is given. We recall the main concepts and results: stratified graph, interpretation of a stratified graph, the valuation mapping and the answer mapping. Several applications are shortly summarized.

Keywords: labeled graph, Peano algebra, morphism, stratified graph, inference, interrogation, interpretation
Math. Subjects Classification 2000: 68T30

1. INTRODUCTION

The concept of *labeled stratified graph* (LSG) was introduced in [2] in connection with that of knowledge base with output. This concept of LSG was applied successfully in various domains and we review such applications in a separate section of this paper.

The structure of the paper is the following: in this Section 1 some basic notions of universal algebra are presented; in Section 2 we recall the following concepts: labeled stratified graph (shortly, stratified graph), accepted path, interpretation, valuation and answer mapping; in Section 3 we give a short description of the domains where the stratified graphs were applied; the last section includes some formulas used in this paper.

We consider a non empty set A. By a *binary partial operation* on A we understand a partial mapping f from $A \times A$ to A. This means that f is defined for the elements of some set $dom(f)$, where $dom(f) \subset A \times A$. We shall use the notation $f : dom(f) \longrightarrow A$. In the case when $dom(f) = A \times A$ we say that f is a *binary operation* on A.

We shall write $f \prec g$ if $f : dom(f) \longrightarrow A$ and $g : dom(g) \longrightarrow A$ are two functions such that $dom(f) \subseteq dom(g)$ and $f(x) = g(x)$ for all $x \in dom(f)$.

By a *partial σ-algebra* we understand a pair ([1]) $\mathcal{A}$=(A, σ_A), where A is the support set of $\mathcal{A}$ and σ_A is *a partial binary operation* on A. If $dom(\sigma_A) = A \times A$ then we say that $\mathcal{A}$ is a *σ-algebra*.

We consider a non-empty set S. If $\rho_1 \in 2^{S\times S}$ and $\rho_2 \in 2^{S\times S}$ then we define $\rho_1 \circ \rho_2$ as the set of all pairs $(x, y) \in S \times S$ for which there is $z \in S$ such that $(x, z) \in \rho_1$ and $(z, y) \in \rho_2$. We introduce the mapping

$$prod_S : dom(prod_S) \longrightarrow 2^{S\times S}$$

where $prod_S(\rho_1, \rho_2) = \rho_1 \circ \rho_2$ and $(\rho_1, \rho_2) \in dom(prod_S)$ if and only if and $\rho_1 \circ \rho_2 \neq \emptyset$

2. STRATIFIED GRAPHS

By a *labeled graph* we understand a tuple $G = (S, L_0, T_0, f_0)$ where S is the set of nodes, L_0 is a finite set of labels, $T_0 \subseteq 2^{S \times S}$ is a set of binary relations on S and $f_0 : L_0 \longrightarrow T_0$ is a surjective mapping. For instance, we can consider the labeled graph G_0 from Figure 1, where we have the set $S = \{x_1, \ldots, x_6\}$ of *nodes* and several *arcs* labeled by $a_1, \ldots, a_6$. We denote by $L_0 = \{a_1, \ldots, a_6\}$ the set of all these labels. If we erase these labels, then the structure obtained from Figure 1 is not a graph because some nodes are connected by two arcs.

We denote by $R(prod_S)$ the set of all restrictions of the mapping $prod_S$:

$$R(prod_S) = \{u \mid u \prec prod_S\}$$

We observe that if u is an element of $R(prod_S)$ then the pair $(2^{S\times S}, u)$ is a partial algebra. This is a partial algebra used to obtain the structure named labeled stratified graph.

Take $u \in R(prod_S)$ and consider the closure $T = Cl_u(T_0)$ of T_0 in the algebra $(2^{S\times S}, u)$.

For each nonempty set M there is a Peano σ-algebra over M. Two Peano σ-algebras are isomorphic algebras and for this reason we shall use the following structure. We consider the set B given by

$$B = \bigcup_{n\geq 0} B_n \qquad (1)$$

where

$$\begin{cases} B_0 = M \\ B_{n+1} = B_n \cup W_n \end{cases} \qquad (2)$$

where $W_n = \{\sigma(x_1, x_2) \mid x_1, x_2 \in B_n\}$ and $\sigma(x_1, x_2)$ is the word $\sigma x_1 x_2$ over the alphabet $\{\sigma\} \cup M$. The pair $PA(M) = (B, \sigma)$ is a Peano σ-algebra over M.

We consider some collection of subsets of B, denoted by $Initial(M)$. Namely, we say that $L \in Initial(M)$ if the following conditions are fulfilled:

- $M \subseteq L \subseteq B$
- if $\sigma(u, v) \in L$ then $u \in L$ and $v \in L$

Generally speaking, if $L \in Initial(M)$ then the pair (L, σ_L), where

- $dom(\sigma_L) = \{(x, y) \in L \times L \mid \sigma(x, y) \in L\}$
- $\sigma_L(x, y) = \sigma(x, y)$ for every $(x, y) \in dom(\sigma_L)$

is a partial σ-algebra.

We consider the Peano σ-algebra $PA(L_0) = (B, \sigma)$ over L_0, where B is given by (1) and (2) for $M = L_0$.

A **labeled stratified graph** $\mathcal{G}$ over G (shortly, **stratified graph** or *LSG*) is a tuple (G, L, T, u, f) where

- $G = (S, L_0, T_0, f_0)$ is a labeled graph
- $L \in Initial(L_0)$
- $u \in R(prod_S)$ and $T = Cl_u(T_0)$
- $f : (L, \sigma_L) \longrightarrow (2^{S\times S}, u)$ is a morphism of partial algebras such that $f_0 \prec f$, $f(L) = T$ and if $(f(x), f(y)) \in dom(u)$ then $(x, y) \in dom(\sigma_L)$

The condition for the morphism f can be graphically represented as in Figure 2, where we observe that we have a commutative diagram.

We denote by $Strat(G)$ the set of all *LSGs* over G. As we proved in [3] we have $Strat(G) \neq \emptyset$. Moreover, we proved in [6] that for every $u \in R(prod_S)$ there is just one stratified graph $\mathcal{G} = (G, L, T, u, f)$ over G and this structure is obtained applying the following steps ([6]):

- Take a labeled graph $G = (S, L_0, T_0, f_0)$
- Take $u \in R(prod_S)$
- Compute $T = Cl_u(T_0)$
- Take $\{B_n\}_{n\geq 0}$ as in (2) for $M = L_0$
- Take $D_0 = L_0$ and define for every natural number $n \geq 0$ the entities (5), (6) and (7).
- Define the mapping $f : dom(f) \longrightarrow T$ as follows:

  $dom(f) = \bigcup_{n\geq 0} dom(f_n) = \bigcup_{k\geq 0} D_k$
  $f(x) = f_k(x)$ if $x \in D_k$, $k \geq 0$
- Take $L = dom(f)$

The inference process performed in a stratified graph is a path based mechanism. An useful concept is that of **structured path** in a stratified graph. We consider a path

$$d = ([x_1, \ldots, x_{n+1}], [a_1, \ldots, a_n]) \quad (3)$$

in a labeled graph $G = (S, L_0, T_0, f_0)$. Consider the least set $STR(d)$ satisfying the following conditions:

- $([x_i, x_{i+1}], a_i) \in STR(d)$, $i \in \{1, \ldots, n\}$
- if $([x_i, \ldots, x_k], b_1) \in STR(d)$ and $([x_k, \ldots, x_r], b_2) \in STR(d)$, where $1 \leq i < k < r \leq n+1$, then $([x_i, \ldots, x_r], [b_1, b_2]) \in STR(d)$

The maximal length elements of $STR(d)$, namely, the elements of the form $([x_1, \ldots, x_{n+1}], c) \in STR(d)$, are called **structured paths** over d. We consider the projection of $STR(d)$ under the second axis:

$$STR_2(d) = \{\beta \mid \exists\alpha : (\alpha, \beta) \in STR(d)\}$$

For example, if in Figure 1 we take the path $d = ([x_1, x_2, x_3, x_4], [a_1, a_2, a_5])$ then only the following structured paths over d can be obtained:

$([x_1, x_2, x_3, x_4], [[a_1, a_2], a_5])$
$([x_1, x_2, x_3, x_4], [a_1, [a_2, a_5]])$

Let d be a path as in (3). We can define the mapping

$$h : STR_2(d) \longrightarrow B$$

as follows:

- $h(x) = x$ for $x \in L_0$
- $h([u, v]) = \sigma(h(u), h(v))$

Thus, $h([a_1, a_2]) = \sigma(a_1, a_2)$, $h([a_1, [a_2, a_5]]) = \sigma(a_1, \sigma(a_2, a_5))$. Some elements of $STR(d)$ are named **accepted structured path**. More precisely, the path $([x_1, \ldots, x_{n+1}], c) \in STR(d)$ is an accepted structured path if $h(c) \in L$.

Because the accepted structured paths are used in the inference process, it is convenient to denote a structured path by $d_s = ([x_1, \ldots, x_{n+1}], h(c))$ instead of $d_s = ([x_1, \ldots, x_{n+1}], c)$.

We denote by $ASP(\mathcal{G})$ the set of all accepted structured paths over $\mathcal{G}$.

Every accepted structured path over $\mathcal{G}$ can be broken into two accepted structured paths over $\mathcal{G}$, as we state in the following proposition:

Proposition 1. ([7]) *For every accepted structured path* $([x_1, \ldots, x_{n+1}], \sigma(v_1, v_2)) \in ASP(\mathcal{G})$, *where* $n \geq 2$, *there is one and only one* $i \in \{2, \ldots, n\}$ *such that* $([x_1, \ldots, x_i], v_1) \in ASP(\mathcal{G})$ *and* $([x_i, \ldots, x_{n+1}], v_2) \in ASP(\mathcal{G})$.

The number i stated in Proposition 1 is named the *break index* for a path d and is denoted by $ind(d)$.

In order to perform an inference, the concepts of **interpretation** and **valuation** are used:

Definition 1 An *interpretation* for $\mathcal{G}$ is a tuple

$$\Sigma = (Ob, i, D, \mathcal{P})$$

where:

• Ob is a finite set of *objects* such that

$$Card(Ob) = Card(S)$$

• $i : S \longrightarrow Ob$ is a bijective mapping

• $D = (Y, *)$ is a partial algebra; Y is called the *domain* of Σ and $*$ is a partial binary operation on Y

• $\mathcal{P} = \{p_a\}_{a \in L_0}$, where

$$p_a : Ob \times Ob \longrightarrow Y$$

Definition 2 The *valuation mapping* generated by Σ is the mapping

$$val_\Sigma : ASP(\mathcal{G}) \longrightarrow Y$$

defined inductively by (8) where $i = ind([x_1, \ldots, x_{n+1}], \sigma(v_1, v_2))$ and $x(i;j) = [x_i, \ldots, x_j]$. Intuitively, the computation can be described as in Figure 3.

Consider a stratified graph $\mathcal{G} = (G, L, T, u, f)$ over $G = (S, L_0, T_0, f_0)$ and $\Sigma = (Ob, i, D, \mathcal{P})$ an interpretation for $\mathcal{G}$. A pair $(x, y) \in S \times S$ is called *interrogation*. For a given interrogation (x, y) we designate by $ASP(x, y)$ the set of all accepted structured paths from x to y in $\mathcal{G}$. The *answer mapping* is the mapping

$$Ans : S \times S \longrightarrow Y \cup \{no\}$$

defined by (9).

The *inference process* generated by $d \in ASP(\mathcal{G})$ is the computation performed to obtain $val_\Sigma(d)$ by (8). The element $val_\Sigma(d)$ is the *conclusion* of the corresponding process.

3. APPLICATIONS OF LSGs

The labelled stratified graphs were used in the following applications:

1. *Semantic of communication.*
2. *Image synthesis.*
3. *Reconstruction a graphical image by extracting the semantics of a linguistic spatial description given in a natural language.*
4. *The modeling of the fusion action for two companies.*
5. *Solving the problems which can be transposed in attribute graphs or colored graphs.*
6. *Knowledge bases with output and their use to the scheduling problems.*

The reader can find the first application in [7]. The second application is treated in detail in the same paper. In connection with this case we relieve the following aspect. We consider the graph represented in Figure 1 and the nodes are interpreted as points in the plan $R \times R$. The space Y is the set of all geometric figures of the plan $R \times R$ such that each figure has an axis of symmetry (a circle, a regular or isosceles triangle etc.

Moreover, we consider the following case: if X and Y are two polygonal lines then $X * Y$ is, in general, the "least" polygon containing both X and Y. We consider the structured paths $d_1 = ([x_1, x_2, x_5, x_6], [[a_1, a_3], a_6])$ and $d_2 = ([x_1, x_2, x_5, x_6], [a_1, [a_3, a_6]])$ for $d = ([x_1, x_2, x_5, x_6], [a_1, a_3, a_6])$. Computing the valuation mapping for the accepted structured path

$$([x_1, x_2, x_5, x_6], \sigma(\sigma(a_1, a_3), a_6))$$

we obtain the image represented in Figure 4.

Now we compute the valuation mapping for the accepted structured path

$$([x_1, x_2, x_5, x_6], \sigma(a_1, \sigma(a_3, a_6)))$$

and obtain the image from Figure 5.

We shall remark that we considered the path

$$d = ([x_1, x_2, x_5, x_6], [a_1, a_3, a_6])$$

and the following accepted structured paths of d:

$$d_1 = ([x_1, x_2, x_5, x_6], \sigma(\sigma(a_1, a_3), a_6))$$

$$d_2 = ([x_1, x_2, x_5, x_6], \sigma(a_1, \sigma(a_3, a_6)))$$

Computing the valuation mapping for these structured paths with respect to some interpretation we obtain different conclusions. Applying (9) we conclude that $Ans(x_1, x_6)$ contains two geometric figures, but $Ans(x_6, x_1) = no$.

In what concerns the third application we mention that in [8] the following problem is treated: *Let's consider we have a remote connection from a node A to a node B. At the node A we have some objects arranged in a spatial area. Through this connection a linguistic description for the relative positions of the objects from A is sent to the node B. The description contains non-fuzzy directional relations. The problem is to reconstruct in B the image from A.* In [8] we proposed a method of hierarchical reasoning about the directional relations received from the node A based on the labeled stratified graphs. If the spatial area from the node A has n-dimensions then, in the reasoning process, we use n labeled stratified graphs, one for each main axis of the area. In connection with this aspect we relieve the following problem treated in [5]: *We suppose we have a remote communication line from S to R. S is the sender and R is the receiver. Let's suppose at the node S we have a chessboard of only* 5×5 *squares and five pawns of the same color. We say that a pawn* P_1 *captures the pawn* P_2 *if* P_1 *and* P_2 *are aligned on the same diagonal of the board at one square distance. We suppose that in S we see an arrangement of these pieces such that no pawn captures any other pawn. The problem is to transmit to R a text description of the scene such that R is able to reconstruct the image from S.* In order to model the communication we used a Recursive Transition Network. We have implemented this case in an application that consists of a Java applet and a Prolog file. In this application we used the JIProlog (JavaInternetProlog) product written by Ugo Chirico ([11]) in order to establish a connection between the Java applet and the Prolog file.

In [4] the following problem is treated: the embedding of two stratified graphs in another stratified graph such that several restrictions are satisfied. This problem is extended to another problem, namely, the collaboration of two or more stratified graphs. This problem was suggested by the research described in [6]. A minimal stratified graph containing the components can be obtained and this structure can be completed such that some additional conditions are satisfied. The model can be applied successfully to describe the actions performed as a result of a fusion of two companies.

In [6] the following problem is treated. We consider a directed graph $G_0 = (S, \Gamma)$, where $S = \{x_1, \ldots, x_m\}$ is the set of its nodes. Let $A = \{a_1, \ldots, a_k\}$ be a set of properties or attributes for the elements of Γ. The attributes $a_1, \ldots, a_k$ may represent colours or other properties such as *closed road, works in progress, rain* and so on.

An *attribute graph* is a tuple $(G_0, A, attr)$, where

- $G_0 = (S, \Gamma)$ is a directed graph
- $A = \{a_1, \ldots, a_k\}$ is the set of attribute names
- $attr : \Gamma \longrightarrow 2^A \setminus \{\emptyset\}$ is a mapping such that $attr(x_i, x_j)$ is the set of all attributes associated to $(x_i, x_j) \in \Gamma$
- for each $i \in \{1, \ldots, k\}$ there is $(x_p, x_q) \in \Gamma$ such that $a_i \in attr(x_p, x_q)$

A *path* in an attribute graph $(G_0, A, attr)$ is a pair

$$([x_1, \ldots, x_n], [a_1, \ldots, a_{n-1}])$$

where $(x_i, x_{i+1}) \in \Gamma$ and $a_i \in attr(x_i, x_{i+1})$ for $i \in \{1, \ldots, n-1\}$.

We consider the mappings $R : A \longrightarrow 2^A$, $K : A \longrightarrow 2^A$ and a subset $N_0 \subseteq A$. By definition, an *accepted path* from y_1 to y_n is a path $([y_1, \ldots, y_n], [b_1, \ldots, b_{n-1}])$ in $(G_0, A, attr)$ such that $b_1 \in A \setminus N_0$ and $b_{i+1} \in R(b_i) \setminus K(b_i)$ for $i \in \{1, \ldots, n-2\}$. The tuple (R, K, N_0) defines *the restrictions* imposed on accepted paths. We shall denote by $Path_{acc}(y_1, y_n)$ the set of the accepted paths from y_1 to y_n.

Given two arbitrary nodes $y, z \in S$ and a natural number r the following problems arise:

P1) Decide whether or not the set $\bigcup_{s=0}^{r} Path_{acc}^{s}(y, z)$ is a non-empty set, where $Path_{acc}^{s}(y, z)$ denotes the set of all accepted paths from y to z that contain exactly s intermediary nodes. In other words, the problem is to decide whether or not there is an accepted path from y to z containing at most r intermediary nodes.

P2) In the affirmative case, find all these paths.

In order to transpose this problem in terms of labeled graphs we take $L_0 = A = \{a_1, \ldots, a_k\}$ and for each $i \in \{1, \ldots, k\}$ we consider the binary relation $f_0(a_i)$ on S defined by:

$$\{(x_p, x_q) \in \Gamma \mid a_i \in attr(x_p, x_q)\} \quad (4)$$

We consider

$$T_0 = \{\rho \mid \exists a_i \in A : f_0(a_i) = \rho\}$$

and we obtain the labeled graph $G = (S, L_0, T_0, f_0)$, where $f_0 : L_0 \longrightarrow T_0$ is the mapping defined by $f_0(a_i) = Q_i$, where Q_i is defined in (4).

The concept of knowledge base with output was introduced in [2] and this concept is based on stratified graph mechanism. Two kinds of computations are performed in a knowledge base with output: syntactic computations and semantic computations. The semantic computations are performed in an output space and some morphism is used. This structure is applied to solve the following problem: *We consider the airports* $x_1, \ldots, x_m$. *The companies* $L_1, \ldots, L_k$ *organize some non-stop flights between these airports. It is known a set* R *of restrictions concerning the continuation of a travel for a passenger. An element of* R *is a rule of the form* $L_i \longrightarrow P_i$*, where* P_i *is a nonempty* *subset of* $\{L_1, \ldots, L_k\}$*. Such rule specifies the following property: if a passenger arrives in some airport using the company* L_i *then he may continue his travel with the company* L *only if* $L \in P_i$*. Give the answer to the following interrogation: given a pair* (x, y) *of airports, is there a flight from* x *to* y *having at most* n *intermediary airports? In the affirmative case find all the solutions.*

4.OPEN PROBLEMS

The following sentences can constitute the subjects for a future research:

• For a stratified graph $\mathcal{G} = (G, L, T, u, f)$ denote by $D_n(\rho)$ the set $\{\alpha \in D_n \mid f_n(\alpha) = \rho\}$ for every $\rho \in T$. Try to find a recursive method to compute $D_n(\rho)$.

• The set $L = \bigcup_{n\geq 0} D_n$ can be infinite. Such cases are presented in [3]. Obtain a necessary and sufficient condition for this case.

• Use the collaboration of several stratified graphs to manage the distributed knowledge.

• Obtain an algebraic method to describe and compute the structured paths in a LSG.

• Try to apply the stratified graphs to model the reasoning by analogy.

• Obtain a comparative study for stratified graphs and semantic schemas ([9], [10])

5.APPENDIX

$$D_{n+1} = \{\sigma(p,q) \in B_{n+1} \setminus B_n \mid p, q \in dom(f_n), (f_n(p), f_n(q)) \in dom(u)\} \quad (5)$$

$$dom(f_{n+1}) = dom(f_n) \cup D_{n+1} \quad (6)$$

$$f_{n+1}(x) = \begin{cases} f_n(x) & \text{if} \quad x \in dom(f_n) \\ u(f_n(p), f_n(q)) & \text{if} \quad x = \sigma(p,q) \in D_{n+1} \end{cases} \quad (7)$$

$$\begin{cases} val_\Sigma([x,y],a) = p_a(i(x), i(y)) \\ \\ val_\Sigma(x(1;n+1), \sigma(v_1,v_2)) = val_\Sigma(x(1;i), v_1) * val_\Sigma(x(i;n+1), v_2) \end{cases} \quad (8)$$

$$\begin{cases} Ans(x,y) = no \quad \text{if } ASP(x,y) = \emptyset \\ \\ Ans(x,y) = \{val_\Sigma(d) \mid d \in ASP(x,y)\} \text{ if } ASP(x,y) \neq \emptyset \end{cases} \quad (9)$$

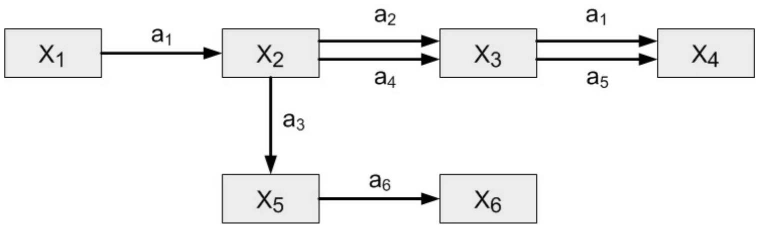

Fig. 1. The labeled graph G_0

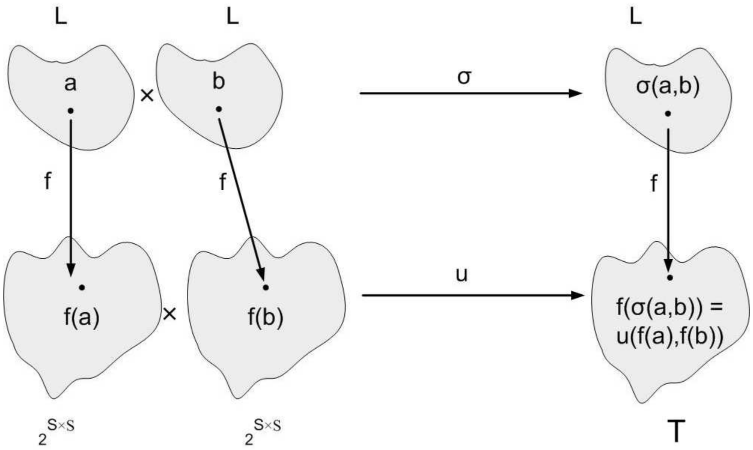

Fig. 2. The morphism condition

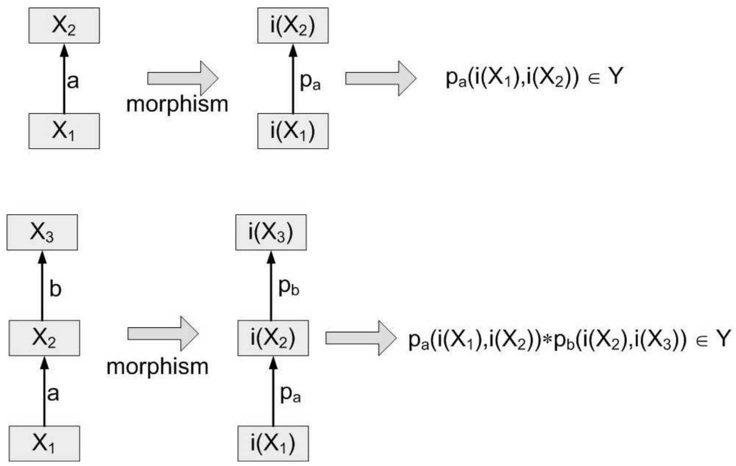

Fig. 3. The valuation mapping

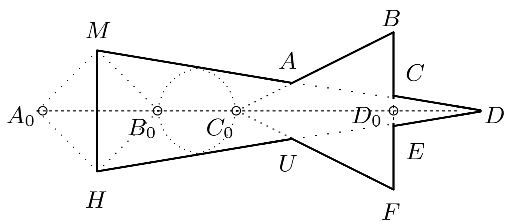

Fig. 4. The first image

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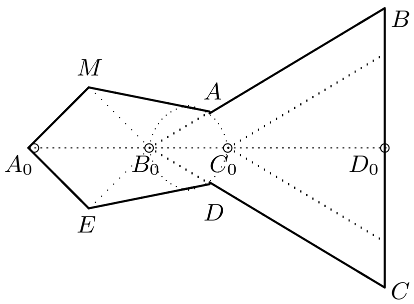

Fig. 5. The second image

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