The Mann Iteration Versus the Ishikawa Iteration - Convergence Analysis

Cristina POPÎRLAN

Faculty of Mathematics and Computer Science,
Department of Computer Science,
University of Craiova, Romania
cristina_popirlan@yahoo.com

Abstract. Suppose we have a real Banach space E and K a nonempty subset of E. Let T be a map on K. This paper analyzes the convergence of the Mann iteration process to a fixed point of T. In the same conditions as above, the convergence of Ishikawa iteration to a fixed point of T is analyzed. The paper compares the fastness of these two convergence when there are imposed condition to T and K.

1 Introduction

Let K be a closed convex subset of a real Banach space E. We consider T a map from K to K with nonempty fixed point set($F(T) \neq \emptyset$).

Definition 1. *The mapping T is called pseudocontractive if and only if*

$$||x-y|| \leq ||(1+t)(x-y) - t(Tx - Ty)||$$

for any $t > 0$ *and for any* $x, y \in K$.

A generalization of pseudocontraction is the strongly pseudocontractive mapping.

Definition 2. *The mapping T is called strongly pseudocontractive if and only if there exists* $c > 1$ *such that*

$$||x-y|| \leq ||(1+t)(x-y) - ct(Tx - Ty)||$$

for any $t > 0$ *and for any* $x, y \in K$.

Let us consider the following two sequences:

1. The sequence $\{x_n\}_{n\geq 0}$ defined by

   $$x_{n+1} = (1-c_n)x_n + c_nTx_n, \text{ with } c_n \in (0,1) \text{ and } x_0 \in K$$

   is called the Mann iteration.
2. The sequence $\{y_n\}_{n\geq 0}$ defined by

$$y_{n+1} = (1 - c_n)y_n + c_nT[(1 - a_n)y_n + a_nTy_n], \text{ with } a_n, c_n \in (0, 1) \text{ and } y_0 \in K$$

is called the Ishikawa iteration.

In some condition the Mann iteration and the Ishikawa iteration are equivalent concerning the convergence to a fixed point of the mapping T. Such a case is analyzed in this paper. From the theorem 2.1 of B.E. Rhoades and S.M. Soltuz from [1], we have that for a strongly pseudocontractive mapping $T : K \to K$, K a closed convex bounded subset of a real space E, for $x_0 = y_0$, the following relations are equivalent:

– Mann iteration converges to a fixed point of T;
– Ishikawa iteration converges to a fixed point of T.

where Mann iteration and Ishikawa iteration are defined earlier.

2 Main result

Definition 3. *$T : K \to K$ is called generalized Lipschitzian mapping if there exists $L > 0$ such that*

$$||Tx - Ty|| \leq L(1 + ||x - y||), \text{ for any } x, y \in K.$$

As an extension to the theorem of Rhoades and Soltuz we have the following theorem:

Theorem 1. *Let E be a real space, K a closed convex subset of E and $T : K \to K$ a strongly pseudocontractive mapping and we define the following two sequences:*

1. $\{x_n\}_{n \geq 0}$ defined by

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, \text{ with } c_n \in (0, 1) \text{ and } x_0 \in K$$

2. $\{y_n\}_{n \geq 0}$ defined by

$$y_{n+1} = (1 - c_n)y_n + c_nT[(1 - a_n)y_n + a_nTy_n], \text{ with } a_n, c_n \in (0, 1) \text{ and } y_0 \in K$$

The following relations are equivalent:

– Mann iteration converges to a fixed point of T;
– Ishikawa iteration converges to a fixed point of T.

Proof
(1) ⇒ (2)
Let x^* be a fixed point of T and let's assume that

$$||x_n - x^*|| \to 0 \text{ when } n \to \infty.$$

We must prove that

$$||y_n - x^*|| \to 0 \text{ when } n \to \infty.$$

For this, we have the following relations:

$$||y_{n+1}-x_{n+1}||^2 = ||(1-c_n)y_n+c_nT[(1-a_n)y_n+a_nTy_n]-(1-c_n)x_n-c_nTx_n||^2 =$$

$$= ||(1-c_n)(y_n-x_n)+c_n(T[(1-a_n)y_n+a_nTy_n]-Tx_n)||^2 \leq$$

$$\leq (1-c_n)^2||y_n-x_n||^2+2c_n\langle T[(1-a_n)y_n+a_nTy_n]-Tx_n, j(y_{n+1}-x_{n+1})\rangle \leq$$

$$\leq (1-c_n)^2||y_n-x_n||^2+2c_n\langle T[(1-a_n)y_n+a_nTy_n]-Tx_n, j(y_n-x_n)\rangle +$$

$$+2c_n\langle T[(1-a_n)y_n+a_nTy_n]-Tx_n, j(y_{n+1}-x_{n+1})-j(y_n-x_n)\rangle \leq$$

$$\leq (1-c_n)^2||y_n-x_n||^2+2c_nk||(1-a_n)y_n+a_nTy_n-x_n||^2+$$

$$+2a_n||T[(1-a_n)y_n+a_nTy_n]-Tx_n||*$$

$$*||j(\tfrac{y_{n+1}-x_{n+1}}{1+||y_n-x_n||})-j(\tfrac{y_n-x_n}{1+||y_n-x_n||})||(1+||y_n-x_n||) \leq$$

$$\leq (1-c_n)^2||y_n-x_n||^2+2a_nk||(1-a_n)y_n+a_nTy_n-x_n||+)$$

$$+2a_nL(1+||(1-a_n)y_n+a_nTy_n-x_n||)A_n(1+||y_n-x_n||$$

where $A_n = ||j(\frac{y_{n+1}-x_{n+1}}{1+||y_n-x_n||})-j(\frac{y_n-x_n}{1+||y_n-x_n||})||$.
Because A_n converges to 0 when $n \to \infty$ we have the equivalent relations:

$$||(1-a_n)y_n+a_nTy_n-x_n||^2 =$$
$$||(1-a_n)(y_n-x_n)+a_n(Ty_n-Tx_n)+a_n(Tx_n-x_n)|| \leq$$

$$\leq (1-a_n)^2||y_n-x_n||^2+2a_n\langle Ty_n-Tx_n, j((1-a_n)y_n+a_nTy_n-x_n)\rangle +$$

$$+2a_n\langle Tx_n-x_n, j((1-a_n)y_n+a_nTy_n-x_n) \leq$$

$$\leq (1-a_n)^2||y_n-x_n||^2+2a_nk||y_n-x_n||^2+$$

$$+2a_nL(1+||y_n-x_n||)||j(\tfrac{(1-a_n)y_n+a_nTy_n-x_n}{1+||y_n-x_n||})-j(\tfrac{y_n-x_n}{1+||y_n-x_n||})||(1+||y_n-x_n||)+$$

$$+2a_n||Tx_n-x_n||||(1-a_n)y_n+a_nTy_n-x_n||$$

So we have the following relation:

$$||y_{n+1}-x_{n+1}||^2 \leq (1-a_n(1-k))||y_n-x_n||^2$$

$$\Rightarrow ||y_n-x_n|| \to 0, n \to \infty.$$

We have the following relations:

$$0 \leq ||y_n - x^*|| = ||y_n - x_n + x_n - x^*||$$

$$0 \leq ||y_n - x_n|| + ||x_n - x^*||$$

$$||y_n - x^*|| \to 0, n \to \infty.$$

(2) ⇒ (1)
If in the relation

$$y_{n+1} = (1 - c_n)y_n + c_n T[(1 - a_n)y_n + a_n T y_n]$$

we replace $a_n = 0$ then we obtain

$$y_{n+1} = (1 - c_n)y_n + c_n T y_n$$

So, if we assume that $\{y_n\}_{n\geq 0}$ converges to a fixed point of T then, for $a_n = 0$, we obtain that $\{x_n\}_{n\geq 0}$ converges to a fixed point of T. So this part is proved.

3 Conclusions

We observe that if the mapping T satisfies the conditions of the above theorem then the sequences generated by Mann iteration and Ishikawa iteration are equivalent in what concern the convergence to a fixed point of T.

If the mapping T is Lipschitz then it is continuous and generalized Lipschitian, so for any Lipschitz mapping the sequences generated by Mann iteration and Ishikawa iteration equivalently converges to a fixed point of T.

The same thing happened when T is bounded.

References

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