Projection Algorithm for Solving the Convex Feasibility Problem

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Abstract. It is a well know problem to find an element in the intersection of a family of closed convex sets with nonempty intersection. This article presents an application for this problem.
Keywords: convex feasibility problem, projection algorithm.
Math.Subject Classification 2000: G.1.5

1 Introduction

Let E be a Banach space. I suppose that in this space I have a family of closed convex sets with nonempty intersection,

$$M_i \in E, i = \overline{1,m}, M = \bigcap_{i=1}^{m} M_i \neq \emptyset.$$

The convex feasibility problem is to find a point in $\bigcap_{i=1}^{m} M_i \neq \emptyset$.

There are two cases:

1. on sets M_i we can make projections that can be explicitly calculated;
2. on sets M_i we can not make projections, but we can make projection on an approximation of those sets M_i.

2 Projection algorithm

Let E be a Banach space with the convex closed subsets $M_i \in E$, $i = \overline{1,m}$ with nonempty intersection $M = \bigcap_{i=1}^{m} M_i \neq \emptyset$.

For every $i = \overline{1,m}$ I define the nonexpansive function $T_i^{(n)} : E \to E$ with the property $M_i \subseteq Fix(T_i^{(n)})$.

Suppose $\alpha_i^{(n)} \in [0, 2]$ and the application:

$$R_i^{(n)} = (1 - \alpha_i^{(n)})I + \alpha_i^{(n)} T_i^{(n)}$$

I have the following notation:

$$A^{(n)} = \sum_{i=1}^{m} \alpha_i^{(n)} R_i^{(n)}.$$

I define an algorithm base on the construction of the following sequence:

$$x_0 \in E$$

$$x_{n+1} = A^{(n)} x_n, \text{ for } n \geq 0.$$

Definition 1. *An algorithm has focus if for every i and for every subsequence* $\{x_{n_k}\}_{k\to\infty}$ *we have:*

$$\{x_{n_k}\}_{k\to\infty} \textit{ converges weakly to } x$$

$$\{x_{n_k} - T_i^{(n_k)}\}_{k\to\infty} \text{ converges strongly to } 0$$

goes to $x \in M_i$

Theorem 1. *Let* $T_i : E \to E$, $i = \overline{1,m}$ *be nonexpansive functions and* $M_i = Fix(T_i)$. *If the sequence* $\{T_i^{(n)}\}_{n\to\infty}$ *converge then the algorithm has focus.*

Theorem 2. *The algorithm has the following proprieties:*

– *if* $x \in E$ *and* $n \geq 0$, *then*

$$||x_n - x||^2 - ||x_{n+1} - x||^2 = \sum_{i<j} \lambda_i^{(n)} \lambda_j^{(n)} \alpha_i^{(n)} \alpha_j^{(n)} ||T_i^{(n)} x_n - T_j^{(n)} x_n||^2 +$$

$$+2 \sum_i \lambda_i^{(n)} \alpha_i^{(n)} \langle x_n - T_i^{(n)} x_n, T_i^{(n)} x_n - v \rangle +$$

$$+ \sum_i \lambda_i^{(n)} \alpha_i^{(n)} [2 - \sum_j \lambda_j^{(n)} \alpha_j^{(n)}] ||x_n - T_i^{(n)} x_n||^2$$

– *if* $x \in M$ *and* $n \geq 0$, *then*

$$||x_n - x||^2 - ||x_{n+1} - x||^2 \geq \sum_i \mu_i^{(n)} ||x_n - T_i^{(n)} x_n||^2$$

– *the sequence* $\{x_n\}_{n\to\infty}$ *is monotone and bounded*
– *if* $n \geq 0$, *then*

$$||x_{n+1} - x_n|| \leq \sum_i \lambda_i^{(n)} \alpha_i^{(n)} ||x_n - T_i^{(n)} x_n||$$

– *the algorithm has focus*
– *if* $IntM \neq \emptyset$ *then the sequence* $\{x_n\}_{n\to\infty}$ *converges to a point in E*
– *if the sequence* $\{x_n\}_{n\to\infty}$ *has a subsequence* $\{x_{n_i}\}_{n_i\to\infty}$, $\lim_{n_i\to\infty} d(x_{n_i}, M) = 0$ *then the sequence* $\{x_n\}_{n\to\infty}$ *converges to a point in M.*

3 The convex feasibility problem

Let x be in E. I note the projection of x on M_i with $P(x,i)$. If $x \in M_i$ then $P(x,i) = x$. Suppose I have i_x so that

$$||x - P(x,i_x)|| = \max_i ||x - P(x,i)||.$$

Let $T : E \to E$ be a function with $Tx = P(x, i_x)$.

To find an element in $\bigcap_{i=1}^{m} M_i$ I must find a fix point of the function T, so $\bigcap_{i=1}^{m} M_i = F(T)$.

Lemma 1. *Let $M_i \in E$, $i = \overline{1,m}$, be a family of closed convex sets with nonempty intersection $Int \bigcap_{i=1}^{m} M_i \neq \emptyset$ and $Int \bigcap_{i=1}^{m} M_i$ is bounded. Suppose that there exists the sequence $\{x_k\}_{k\to\infty} \in E$ with $\lim_{k\to\infty} d(x_k, M_i) = 0$ for every $i = \overline{1,m}$.*

Then $\lim_{k\to\infty} d(x_k, \bigcap_{i=1}^{m} M_i) = 0$.

Demonstration

Suppose it exists $o \in Int \bigcap_{i=1}^{m} M_i$. Then exists a closed ball $D(o,r) = \{x \in E, ||x|| \leq r\} \subset Int \bigcap_{i=1}^{m} M_i$. Fie $\epsilon \in (0,1)$ and $C \in R$ with $||x|| \leq C - \epsilon$.

Because $\lim_{k\to\infty} d(x_k, M_i) = 0$, then exists a sequence $\{y_k^{(i)}\}_{k\to\infty} \in M_i$ with the property
$\lim_{k\to\infty} ||y_k^{(i)} - x_k|| = 0$.

I have the following notation:

$$z_k = (1 - \tfrac{C}{\epsilon})(y_k^{(i)} - x_k),\ k \geq 0.$$

Then exists the number $k_i(\epsilon)$ with the property

$$||y_k^{(i)} - x_k|| \leq \tfrac{r}{|1-\frac{C}{\epsilon}|} \text{ for every } k \geq k_i(\epsilon).$$

So $||z_k|| \leq r$, so $z_k \in M_i$.

It follows the next relation:

$$(1 - \tfrac{\epsilon}{C})x_k = \tfrac{\epsilon}{C} z_k + (1 - \tfrac{\epsilon}{C}) y_k^{(i)} \text{ for every } k \geq k_i(\epsilon).$$

So $(1 - \frac{\epsilon}{C})x_k \in M_i$ for every $k \geq k_i(\epsilon)$.

Let $k_0(\epsilon) = \max_i k_i(\epsilon)$. So $(1 - \frac{\epsilon}{C})x_k \in M_i$.

For every $k \geq k_0(\epsilon)$ the next relations are true

$$(1 - \tfrac{\epsilon}{C})x_k \in \bigcap_{i=1}^{m} M_i$$

$$d(x_k, \bigcap_{i=1}^{m} M_i) \le ||x_k - (1-\frac{\epsilon}{C})x_k|| = \frac{\epsilon}{C-\epsilon}||(1-\frac{\epsilon}{C})x_k|| < \epsilon.$$

For the next theorem I have the sequence $\{x_k\}_{k\to\infty} \in E$ obtained with the Mann iteration:

$$x_{k+1} = T_t(x_k),\ T_t = (1-t)I + tT,\ T(x) = P(x, i_x).$$

Theorem 3. *Let $M_i \in E$, $i = \overline{1,m}$, be a family of closed convex sets with nonempty intersection Int $\bigcap_{i=1}^{m} M_i \neq \emptyset$ and Int $\bigcap_{i=1}^{m} M_i$ is bounded. Suppose there exists the sequence $\{x_k\}_{k\to\infty} \in E$.*

Then the sequence $\{x_k\}_{k\to\infty}$ converges strongly to a fix point from $\bigcap_{i=1}^{m} M_i$ for $x_0 \in E$.

Demonstration

Because $F(T_t) = \bigcap_{i=1}^{m} M_i$ is a closed set, T_t is a quasinonexpansiv function and $\lim_{k\to\infty} d(x_k, \bigcap_{i=1}^{m} M_i) = 0$, and using the lemma I obtain that the sequence $\{x_k\}_{k\to\infty}$ converges stongly to a fix point from $\bigcap_{i=1}^{m} M_i$.

4 Application

This application solves a linear system with 4 inequations and 2 unknowns. I consider the following linear system of inequations:

$$a_{11}x + a_{12}y + b_1 > 0$$

$$a_{21}x + a_{22}y + b_2 > 0$$

$$a_{31}x + a_{32}y + b_3 > 0$$

$$a_{41}x + a_{42}y1 + b_4 > 0$$

Every inequation is represented in the application as a line.

1. first all the four lines must be drowned and saved;
2. related to a point on the screen, the surface that interesse us must be selected, this must be done because the application solves a linear system on inequations;
3. then the application determine all the intersections between this lines and then it presents the solution of the system;
4. the user can draw a point on the screen and the application presents the sequence as in the algorithm and the limit of the sequence.

Here I present some examples of this application:

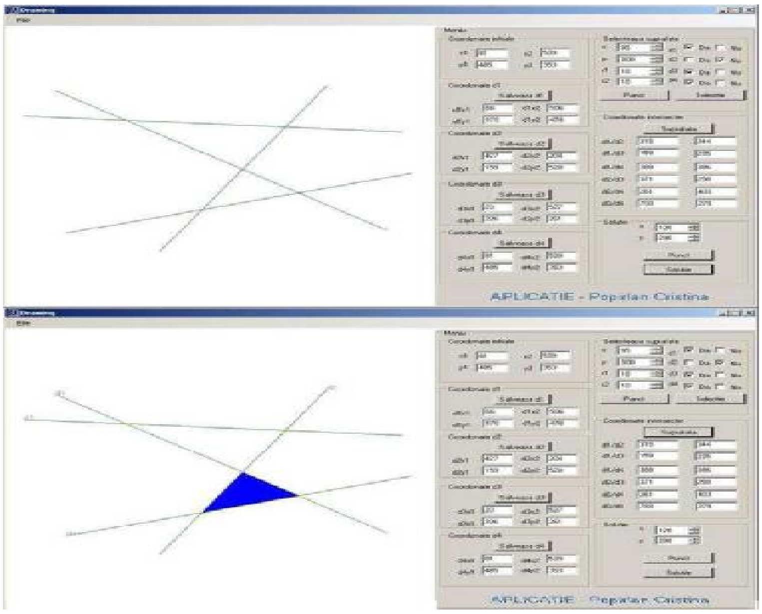

Figure 1. Example 1

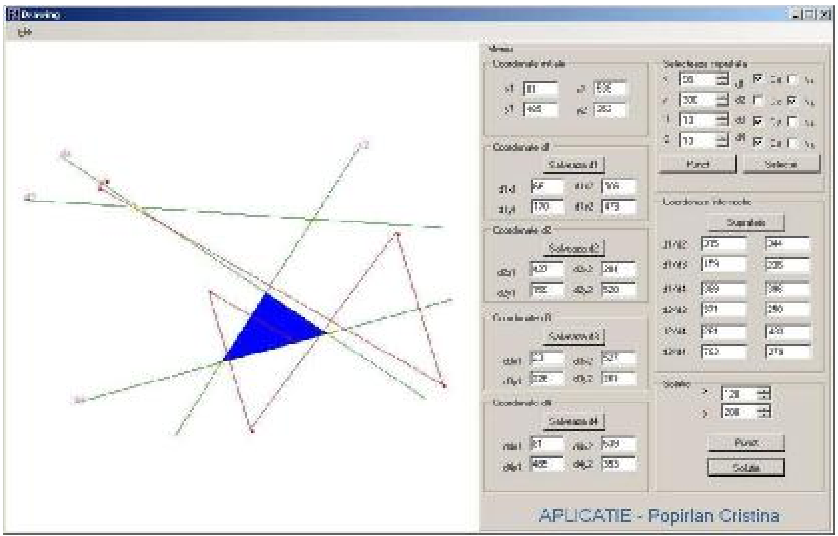

Figure 2. Example 1

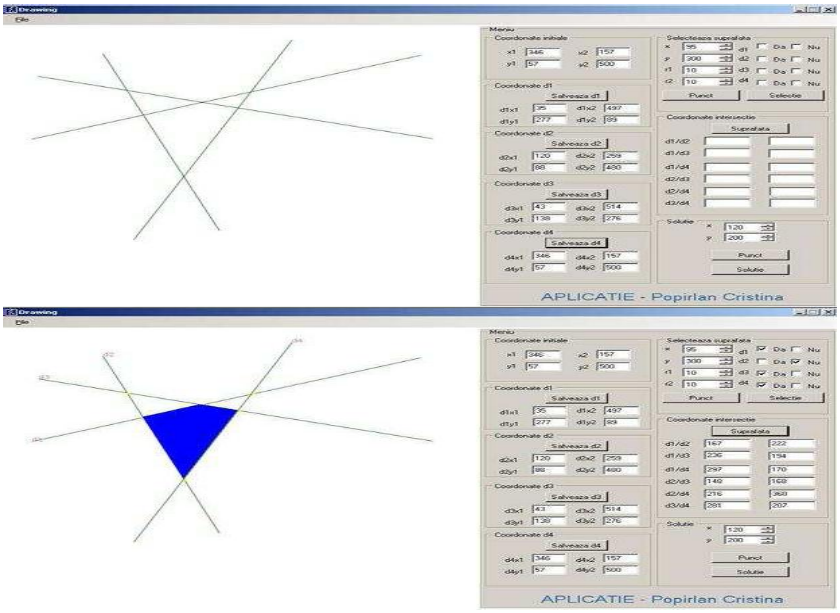

Figure 3. Example 2

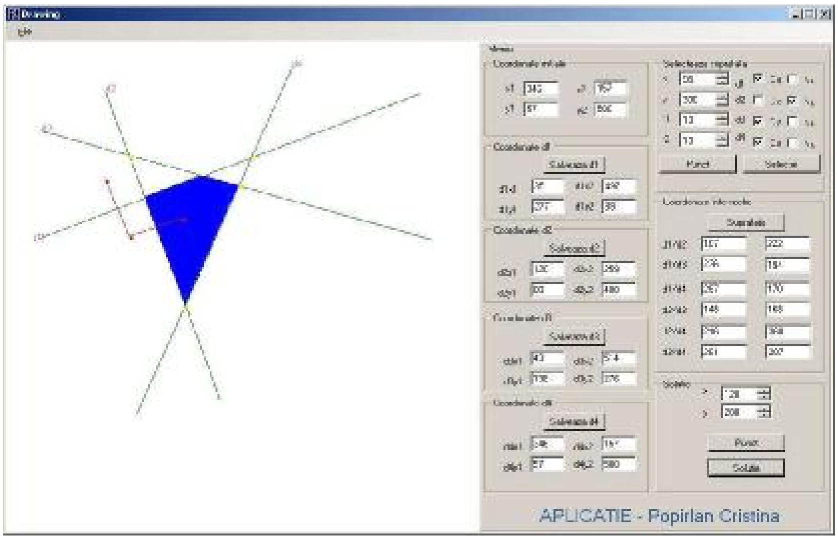

Figure 4. Example 2

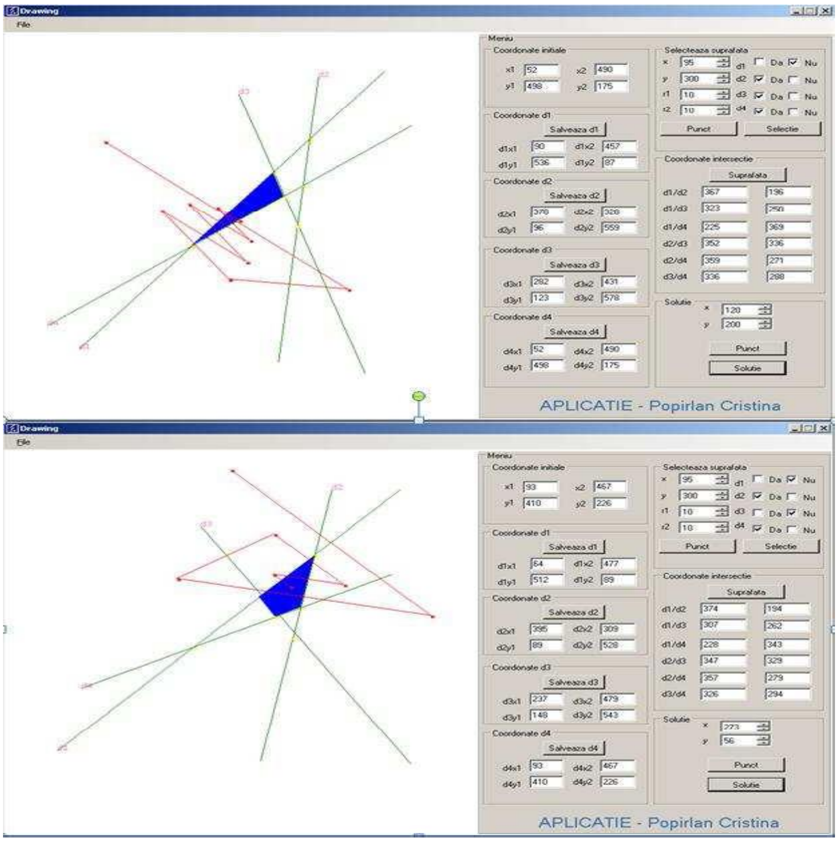

Figure 5. Examples

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