A Numerical Algorithm for a Nonlinear System with Partial Differential Equations of First Order

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Abstract. In this paper we present a numerical study of an system with partial differential equations of first order. The paper contains the theorems of existence and uniqueness of the solution of such a system and an algorithm for approximating the solution with a given error.
Keywords: differential equations, numerical approximation, integral equations
Math. Subject Classification 2000: 65M55

Let:
$D = [a,b] \times [c,d] \subset \mathbf{R}^2$
$(x_0, y_0) \in D$
$u_0 \in \mathbf{R},\ S = [u_0 - r, u_0 + r],\ r \in (0,\infty)$
and the functions $f, g : D \times S \to \mathbf{R}$ which have the following properties:
(a) $f, g \in C^1(D \times S)$
(b) $\dfrac{\partial f}{\partial y}(x,y,u) + \dfrac{\partial f}{\partial u}(x,y,u)g(x,y,u) = \dfrac{\partial g}{\partial x}(x,y,u) + \dfrac{\partial g}{\partial u}(x,y,u)f(x,y,u),$
for $(x,y,u) \in D \times S$.

We consider the nonlinear system with partial differential equations of first order:

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,y) = f(x,y,u(x,y)) \\ \dfrac{\partial u}{\partial y}(x,y) = g(x,y,u(x,y)) \end{cases}, (x,y) \in D \tag{1}$$

so that:

$$u(x_0, y_0) = u_0 \tag{2}$$

Theorem 1. *The function $u : D \to \mathbf{R}$ is solution on D for the problem* (1)&(2) *if and only if u is a continue solution on D for the integral equation :*

$$u(x,y) = u_0 + \int_{x_0}^{x} f(s,y,u(s,y))ds + \int_{y_0}^{y} g(x_0,t,u(x_0,t))dt \tag{3}$$

Let:

$$\|f\| = \sup\{|f(x,y,u)|, (x,y,u) \in D \times S\}$$
$$\|g\| = \sup\{|g(x,y,u)|, (x,y,u) \in D \times S\}$$
$$M = \max\{\|f\|, \|g\|\}$$
$$h = \min\left\{\min\{b - x_0, x_0 - a\}, \frac{r}{2M}\right\}$$
$$k = \min\left\{\min\{d - y_0, y_0 - c\}, \frac{r}{2M}\right\}$$
$$\varPi = [x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k]$$

Theorem 2. *(1)The sequence of real functions* $(u_q)_q$ *defined on the rectangle* $\varPi$ *as follows:*

$$\begin{cases} u_0(x,y) = u_0 \\ u_{p+1}(x,y) = u_0 + \int_{x_0}^{x} f(s,y,u_p(s,y))ds + \int_{y_0}^{y} g(x_0,t,u_p(x_0,t))dt,\ p \geq 0 \end{cases} \tag{4}$$

is uniformly convergent on the rectangle $\varPi$ *and its limit* u *is a solution of* (1)&(2)*;*
(2)The Problem (1)&(2) *has a single solution* $u : \varPi \to \mathbf{R}$.

For the numerical solving of the problem (1)&(2) or, equivalent, of the integral equation (3), we will propose the following algorithm which consist in:
(i) The discreteness of the rectangle $\varPi$ determined of the points:

$$\begin{cases} \xi_i = x_0 - h + (i-1)\dfrac{h}{m},\ 1 \leq i \leq 2m+1 \\ \eta_j = y_0 - k + (j-1)\dfrac{k}{n},\ 1 \leq j \leq 2n+1 \end{cases} \tag{5}$$

(ii) The use of a numerical integration method (*Step* 4.1) and an algorithm for the numerical convergence acceleration (the Richardson algorithm, *Step* 4.4) for the integrals on the equality:

$$u_{p+1}(\xi_i,\eta_j) = u_0 + \int_{\xi_{m+1}}^{\xi_i} f(s,\eta_j,u_p(s,\eta_j))ds + \int_{\eta_{n+1}}^{\eta_j} g(\xi_{m+1},t,u_p(\xi_{m+1},t))dt \tag{6}$$

The calculus finishes when:

$$\max_{\substack{1\leq i\leq 2m+1 \\ 1\leq j\leq 2n+1}} \mid u_{p+1}(\xi_i,\eta_j) - u_p(\xi_i,\eta_j) \mid < \varepsilon,$$

where ε is admissible error.

For $m, n \in \mathbb{N}^*$ the algorithm is the following:

$$
\begin{array}{l}
Step\ 1: \ m_0 = m,\ n_0 = n \\
Step\ 2: \ For\ i = 1,2,...,2m+1: \\
\qquad \xi_i = x_0 - h + (i-1)\dfrac{h}{m} \\
\quad For\ j = 1,2,...,2n+1: \\
\qquad \eta_j = y_0 - k + (j-1)\dfrac{k}{n} \\
Step\ 3: \ For\ i = 1,2,...,2m+1: \\
\quad For\ j = 1,2,...,2n+1: \\
\qquad u_0(\xi_i,\eta_j) = u_0 \\
Step\ 4: \ For\ p \geq 0: \\
\quad Step\ 4.1: \ For\ i = 1,2,...,2m+1: \\
\qquad For\ j = 1,2,...,2n+1: \\
\qquad\quad u_{p+1}(\xi_i,\eta_j) = u_0 + S_{1p}(\xi_i,\eta_j) + S_{2p}(\xi_{m+1},\eta_j) \\
\quad Step\ 4.2: \ For\ i = 1,2,...,2m+1: \\
\qquad For\ j = 1,2,...,2n+1: \\
\qquad\quad w_0(i,j) = u_{p+1}(\xi_i,\eta_j) \\
\quad Step\ 4.3: \ For\ s = 1,2,...,r,\ r \in \mathbb{N}^*: \\
\qquad\quad m = 2m \\
\qquad\quad n = 2n \\
\qquad\quad Step\ 2 \\
\qquad\quad Step\ 4.1 \\
\qquad\quad For\ i = 1,2,...,2m_0+1: \\
\qquad\quad For\ j = 1,2,...,2n_0+1: \\
\qquad\qquad w_s(i,j) = u_{p+1}(\xi_{2^s(i-1)+1},\eta_{2^s(j-1)+1}) \\
\quad Step\ 4.4: \ m = m_0 \\
\qquad n = n_0 \\
\qquad Step\ 2 \\
\qquad For\ i = 1,2,...,2m+1: \\
\qquad For\ j = 1,2,...,2n+1: \\
\qquad\quad Apply\ the\ Richardson\ algorithm\ for: \\
\qquad\quad w_0(i,j), w_1(i,j), ..., w_r(i,j) \\
\qquad\quad u_{p+1}(\xi_i,\eta_j) = R_r^{(0)}
\end{array}
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